REAL-VALUED FUNCTIONS ON FLOWS

RICHARD N. BALL AND JAMES N. HAGLER

ABSTRACT. We develop the flow analog of the classical Yosida adjunction between spaces and archimedean lattice-ordered groups with strong unit. A product of this development is the flow counterpart of the classical compactification of a space. We characterize those flows which are compactifiable, i.e., dense subflows of a compact flow. Finally, we exhibit a duality between the compactifications of a given flow and the topologies on the monoid of actions.

1. INTRODUCTION

In order to understand the dynamics of a flow it is sometimes helpful to enlarge it. But such an approach can be most penetrating only within the context of a general and systematic framework. It is the intent of this article to initiate such an analysis, at least in the case of flow compactifications; to be more precise, we develop the flow counterparts of the theory of compactifications. But in truth we have a broader agenda in mind, namely to apply the machinery of general topology to topological dynamics. In fact, this article is the second of a sequence with this theme: in the first [4] we develop the flow analog of the absolute (a.k.a. Gleason cover), and in [5] we develop Stone duality. (See also [6] and [7] for related work.)

Throughout this article T denotes a topological monoid, whose members we call *actions* and denote by t or s, sometimes with subscripts. We use $\mathcal{N}_t$ to denote the neighborhood filter of t.

Definition . *We say that* the topology of T is based at 1 *if* $T_1 t \in \mathcal{N}_t$ *for all* $t \in T$ *and* $T_1 \in \mathcal{N}_1$.

This condition means that the right translates of the neighborhoods of the identity form a base for the neighborhoods of t. It holds, for

---

Date: November 25, 1997.

1991 *Mathematics Subject Classification.* Primary 54H20; Secondary 06F20, 34C35.

Key words and phrases. flow, lattice-ordered group, Stone-Čech compactification.

example, in any topological group. We say that T *acts on a space* X if there is a monoid homomorphism $\phi_X : T \to \hom_{\mathbf{Sp}}(X, X)$, where **Sp** denotes the category of spaces with continuous maps. That is,

- $\phi_X 1$ is the identity function on X, and
- $\phi_X(t_1 t_2) = (\phi_X t_1)(\phi_X t_2)$ for all $t_i \in T$.

We systematically suppress nearly all mention of ϕ_X, writing $(\phi_X t)\, x$ as simply tx. Note that T need not act faithfully on X; if, for example, ϕ_X takes each action to 1_X then we say that T *acts trivially on* X. A subspace $Y \subseteq X$ is T*-invariant* if $ty \in Y$ for all $y \in Y$ and $t \in T$.

A *flow* is a space X on which T acts in such a way as to make the evaluation map $(t, x) \mapsto tx$ continuous. A *flow map* $f : X \to Y$ is a continuous function between flows which commutes with the actions, i.e., one which makes this diagram commute. A T-invariant subspace of a flow is sometimes called a *subflow.* There is a vast literature on flows; basic references in the spirit of this article are [2], [9], [10], and [18].

$$\begin{tikzcd}
X \arrow[r, "t"] \arrow[d, "f"'] & X \arrow[d, "f"] \\
Y \arrow[r, "t"'] & Y
\end{tikzcd}$$

We briefly describe the organization of this article. Section 2 contains an introduction to lattice ordered groups, hereafter referred to simply as *ℓ-groups*, and to the notion of the action of a monoid T on an ℓ-group G. In Section 3 explicit ℓ-group conditions are given which are equivalent to the action of T on G being continuous. T-pseudometrics and T-uniformly continuous functions are defined and their properties examined in Section 4. Sections 5 and 6 contain the developments of Yosida duality and compactifications for flows.

Up to this point, our results on flow compactifications are analogous to those for space compactifications and much of the extension to flows is routine. The situation is made more interesting by the fact that the analogy is not complete. In a sense, the new mathematics begins only in Section 7; the reader may wish to skip to this section and refer to earlier sections for definitions and statements of results.

In order to illustrate where these theories diverge, we present two examples in Section 7. Most important, we show (Example 2) that, even for T a topological group and X a separable metric space which is a flow under the actions of T, the flow compactification of X may fail to exist. These examples show the sharpness of the results in the final sections of this paper.

Section 8 contains the development the one-point compactification for flows. We show in this section that if T is a topological group and if X is a locally compact flow then X admits a one-point flow compactification. (Example 1 shows that not every locally compact flow has a one-point flow compactification.) In order to develop the flow counterpart of normality in Section 10, we introduce the concept of a T-scale in Section 9. Finally, Section 11 compares compactifications of X and topologies on the monoid T.

The authors are grateful to the referee for his detailed reading of the manuscript and for his many helpful comments.

2. ℓ-GROUPS

A *lattice-ordered group* is a structure of the form $(G, +, 0, \vee, \wedge)$, where $(G, +, 0)$ is a group, $(G, \vee, \wedge)$ is a lattice, and the group and lattice operations are required to be compatible in the sense that for all $g_i \in G$,

$$g_1 \leq g_2 \Longrightarrow g_1 + g_3 \leq g_2 + g_3 \text{ and } g_3 + g_1 \leq g_3 + g_2.$$

A lattice ordered group is termed an *ℓ-group* in the literature; basic references are [8], [16], [1], and [12]. The *positive cone* of an ℓ-group G is $G^+ = \{g \in G : g \geq 0\}$. Every element of an ℓ-group G is uniquely expressible as the difference of disjoint elements, meaning elements whose infimum is 0. In particular, $g = g^+ - g^-$, where $g^+ = g \vee 0$ and $g^- = (-g) \vee 0$ are the *positive and negative parts of* g, respectively. The *absolute value of* g is $|g| = g^+ \vee g^- = g \vee (-g)$. An ℓ-group G is said to be *archimedean* if for all $g_i \in G^+$, $ng_1 \leq g_2$ for all $n \in \mathbb{N}$ forces $g_1 = 0$. An archimedean ℓ-group is necessarily abelian. A *weak unit* of G is an element $1 > 0$ such that $1 \wedge g > 0$ for all $0 < g \in G$. A *strong unit* of G is an element $1 > 0$ such that for every $g \in G^+$ there is some positive integer n such that $g \leq n$. (We use n to denote $n1$. Despite the use of 1 to denote the unit, G carries no multiplicative structure.) An *ℓ-homomorphism* is a mapping between ℓ-groups which is simultaneously a group and lattice homomorphism.

An *ℓ-subgroup* of an ℓ-group G is a subset which is simultaneously a subgroup and sublattice of G. An *ℓ-ideal* is a subgroup and sublattice $K \subseteq G$ which is *convex*, i.e., $0 \leq g \leq k \in K$ implies $g \in K$. Note that a proper ℓ-ideal cannot contain a strong unit. This fact is important in what follows because it can be used to produce ℓ-homomorphisms from an abstract abelian ℓ-group G with strong unit 1 into the real numbers. That is because in such an ℓ-group any proper ℓ-ideal is contained in a maximal ℓ-ideal by Zorn's Lemma, and every maximal ℓ-ideal is the kernel of a unique ℓ-homomorphism into the real numbers which takes the strong unit to the real number 1 [8].

The motivating examples of archimedean ℓ-groups with weak unit are those of the form CX, the set of continuous real-valued functions on the space X [16]. The operations on CX are pointwise: for $x \in X$,

$$\begin{aligned}(g_1 + g_2)\,x &= g_1x + g_2x \quad \text{and}\\ (g_1 \vee g_2)\,x &= g_1x \vee g_2x.\end{aligned}$$

Corr 1 When a unit is designated, we always choose the constant function 1 as a weak unit of CX or of any of its ℓ-subgroups. If G is an ℓ-subgroup of CX we denote the zero and cozero sets of an element $g \in G$ as follows.

$$\begin{aligned}\operatorname{zero} g &= \{x \in X : gx = 0\}\,,\\ \operatorname{coz} g &= \{x \in X : gx \neq 0\}\,.\end{aligned}$$

We say that G *separates the points of* X if for all $x_1 \neq x_2$ in X there is some $g \in G$ such that $gx_1 = 0$ and $gx_2 = 1$. We say that G *determines the topology on* X if $\{\operatorname{coz} g : g \in G\}$ forms a base for the topology of X. Note that G separates the points whenever it determines the topology on a T_1 space X (the T_1 here refers to a separation axiom, not a monoid of actions), but not conversely, unless X is compact.

In this article we are interested in the smaller ℓ-group C^*X of *bounded* continuous real-valued functions on the space X, in which 1 is a strong unit. Thus the context for the algebraic portion of this article is the following category.

Definition . *The category* **lGp** *has objects* $(G, +, 0, 1, \vee, \wedge)$*, where* $(G, +, 0, \vee, \wedge)$ *is an archimedean ℓ-group with designated strong unit* 1*. The morphisms of* **lGp** *are the ℓ-homomorphisms which take the strong unit of the domain to the strong unit of the codomain.*

The generalization of the results of this article to the larger ℓ-group CX is straightforward for the most part, but it raises issues (chiefly concerning the domains of reality of the functions) which are irrelevant to our main purpose, that purpose being to lay out the Yosida adjunction for flows. *Therefore we abuse standard terminology in this article by using CX to denote the smaller ℓ-group of bounded continuous real-valued functions on the space X, by calling* **lGp** *objects ℓ-groups, calling* **lGp** *subobjects ℓ-subgroups (they must contain the designated strong unit), and calling* **lGp** *morphisms ℓ-homomorphisms (they must preserve the strong unit).* Adherence to this convention has the consequence that no proper ℓ-ideal is an ℓ-subgroup, since such an ideal can never contain the designated strong unit.

Definition . T acts on the ℓ-group G *if there is a monoid antimorphism* $\phi_G : T \to \hom_{\mathbf{lGp}}(G, G)$*, i.e.,* ϕ_G *satisfies the following requirements.*

- $\phi_G 1$ *is the identity* ℓ*-homomorphism on* G*, and*
- $\phi_G(t_1 t_2) = (\phi_G t_2)(\phi_G t_1)$*.*

If T *acts on the* ℓ*-group* G *then a* $T\ell$-subgroup of G *is an* ℓ*-subgroup* H *of* G *which is closed under the actions of* T*, i.e.,* $ht \in G$ *for all* $h \in H$ *and* $t \in T$*.*

When T acts on an abstract ℓ-group G we usually suppress explicit mention of ϕ_G, writing $(\phi_G t)\, g$ as simply gt. (In this article we generally write functions to the left of their inputs. The reason for the exception in this case is that if X is a space on which T acts and $G = CX$ then each action t on X induces a corresponding action on G by the rule

$$((\phi_G t)\, g)\, x = gtx$$

for $t \in T$ and $x \in X$. T acts on G by this rule, i.e., ϕ_G is a monoid antimorphism, and the association of actions on spaces with actions on their ℓ-groups of continuous functions is canonical; see the definition prior to Corollary 11.)

3. Continuity of evaluation

Definition . *If* T *acts on the* ℓ*-group* G *then for* $g \in G^+$ *and* $t \in T$ *we let*

$$G(g,t) = \left\{ h \in G^+ : \exists T_t \in \mathcal{N}_t\ \forall t' \in T_t\ \ (h \le gt') \right\}.$$

Lemma 1. *Suppose the topology on* T *is based at* 1*. Then in any* ℓ*-group* G *acted upon by* T *we have*

$$G(g, t_1)\, t_2 \subseteq G(g, t_1 t_2).$$

Proof. Consider $h \in G(g, t_1)$, say $h \le g t_1'$ for all t_1' in some neighborhood T_1 of t_1. By the continuity of multiplication in T there is some neighborhood T' of 1 such that $T' t_1 \subseteq T_1$, and $T' t_1 t_2$ is a neighborhood of $t_1 t_2$ because the topology is based at 1. But for any $t' \in T'$ we can write $t' t_1$ as $t_1' \in T_1$, so that $g t' t_1 t_2 = g t_1' t_2 \ge h t_2$. That is, $h t_2 \in G(g, t_1 t_2)$. □

Theorem 2. *The following are equivalent for a Tychonoff space* X *on which* T *acts.*

(1) X *is a flow, i.e., the evaluation map* $(t, x) \longmapsto tx$ *is continuous,*

(2) *If G is a $T\ell$-subgroup of CX which determines the topology on X then for all $g \in G^+$, $t \in T$, and $x \in X$,*

$$gtx = \bigvee_{G(g,t)} hx.$$

(3) *There is some $T\ell$-subgroup G of CX which determines the topology on X such that for all $g \in G^+$ and $t \in T$,*

$$\operatorname{coz} gt = \bigcup_{G(g,t)} \operatorname{coz}\, h.$$

If the topology of T is based at 1 it is enough to verify these conditions at $t = 1$.

Proof. Suppose that evaluation is continuous, and consider $g \in G^+$, $t \in T$, and $x \in X$ such that $x \in \operatorname{coz} gt$. Let ϵ satisfy $0 < \epsilon < gtx$, so that $tx \in \operatorname{coz}(g - \epsilon)^+$. Use the continuity of evaluation to find neighborhoods T_1 of t and X_1 of x such that $t_1x_1 \in \operatorname{coz}(g - \epsilon)^+$ for all $t_1 \in T_1$ and $x_1 \in X_1$. Use the fact that G determines the topology on X to find $h \in G^+$ such that $x \in \operatorname{coz}\, h \subseteq X_1$, and by replacing h by $nh \wedge \epsilon$ for sufficiently large n, we may assume with no loss of generality that $h \leq \epsilon$ and that $hx = \epsilon$. We claim that $h \in G(g, t)$ because $h \leq gt_1$ for all $t_1 \in T_1$. This is so because for any $x_1 \in X$ we have

$$\begin{aligned} 0 < hx_1 &\implies x_1 \in \operatorname{coz}\, h \subseteq X_1 \implies \\ t_1x_1 \in \operatorname{coz}(g - \epsilon)^+ &\implies gt_1x_1 > \epsilon \geq h(x_1). \end{aligned}$$

This shows that $h \in G(g, t)$, and since $hx = \epsilon$ it also shows that $gtx = \bigvee_{G(g,t)} hx$.

Condition (3) follows from (2) by taking $G = CX$, since CX determines the topology on X because X is Tychonoff. Now assume (3) and consider $t \in T$, $x \in X$, and open set U containing tx. Since G determines the topology on X we can find $g \in G^+$ such that $tx \in \operatorname{coz}\, g \subseteq U$. Then

$$x \in \operatorname{coz} gt = \cup_{G(g,t)} \operatorname{coz}\, h$$

implies $x \in \operatorname{coz}\, h$ for some $h \in G(g, t)$; let T_1 be a neighborhood of t such that $h \leq gt_1$ for all $t_1 \in T_1$. Then for any $x_1 \in \operatorname{coz}\, h$ and $t_1 \in T_1$ we have $0 < hx_1 \leq gt_1x_1$, which is to say that $t_1x_1 \in \operatorname{coz}\, g \subseteq U$. This shows that (1) holds.

Finally assume the topology on T is based at 1, and suppose that

$$gx = \bigvee_{G(g,1)} hx$$

for all $g \in G^+$ and $x \in X$. Fix $t \in T$. By replacing x by tx and using Lemma 1 we get

$$gtx = \bigvee_{h \in G(g,1)} htx = \bigvee_{k \in G(g,1)t} kx \leq \bigvee_{k \in G(g,t)} kx \leq gtx,$$

which completes the proof. □

Corollary 3. *If either condition (2) or (3) of Theorem 2 holds for one $T\ell$-subgroup of CX which determines the topology on X then both conditions hold for all such subgroups.*

Recall that a subspace Y is said to be C^*-*embedded* in a space X if every bounded continuous real-valued function on Y is the restriction of such a function on X [11].

Corollary 4. *Let X be a Tychonoff space acted upon by T, let G be CX, and let Y be the subspace of X consisting of those points y which satisfy*

$$gt_1t_2y = \bigvee_{G(g,t_1)} ht_2y$$

for all $g \in G$ and $t_i \in T$. Then Y is a flow, and if X has a dense C^*-embedded T-invariant subspace which is a flow then it has a largest one and it is Y. Corr 3

Proof. It is clear that $y \in Y$ implies $ty \in Y$ for all $t \in T$, i.e., that Y is T-invariant. Furthermore Y is a flow by Theorem 2, since the restrictions of the functions of G to Y form a $T\ell$-subgroup of CY which determines the topology of Y. Now suppose Z is a dense C^*-embedded T-invariant subspace of X which is a flow. Then the bounded continuous real-valued functions on Z extend uniquely to X, so that the fact that Z is a flow implies by Theorem 2 that

$$gt_1z = \bigvee_{G(g,t_1)} hz$$

for all $g \in G^+$, $t_1 \in T$, and $z \in Z$. But replacing z by t_2z gives the condition which defines the membership of z in Y. Finally, since $Y \supseteq Z$ and Z is C^*-embedded, so is Y. □

We do not know if the subspace Y of Corollary 4 is always C^*-embedded. Also, it can easily happen that a compact Hausdorff space X has this subspace Y dense and C^*-embedded, yet X itself is not a flow. In fact, this article can be viewed as an examination of the circumstances in which this situation arises; in particular, Example 1 is of exactly this type.

We would like a condition on an abstract ℓ-group G which is equivalent to being a $T\ell$-subgroup of CX for some flow X. Perhaps it is not surprising that we have no such condition, since abstract ℓ-groups are better modeled by continuous functions on locales than by continuous functions on spaces [3]. If X is compact, however, a particularly appealing ℓ-group condition captures this attribute; see Theorem 20. The next section develops the machinery necessary to analyze this condition.

4. T-UNIFORMLY CONTINUOUS FUNCTIONS

Definition . *Let X be a flow. A T-*pseudometric *on X is a continuous function $d : X \times X \to \mathbb{R}^+$ such that d is a pseudometric with the feature that for all $t \in T$ and all $\epsilon > 0$ there is some $T_t \in \mathcal{N}_t$ satisfying $d(tx, t'x) < \epsilon$ for all $x \in X$ and $t' \in T_t$.*

If the topology on T is based at 1 then the last mentioned feature of a T-pseudometric need only be verified at $t = 1$.

Theorem 5. *Suppose d is a T-pseudometric on the flow X. For each $n \in \mathbb{N}$ let*

$$U_n = \left\{ (x, y) \in X^2 : d(x, y) < \frac{1}{3^n} \right\}.$$

Then each U_n has the following properties.

1. *U_n is an open subset of $X \times X$ containing the diagonal Δ.*
2. *$U_n = U_n^{-1}$.*
3. *$U_{n+1} \circ U_{n+1} \circ U_{n+1} \subseteq U_n$.*
4. *For all $t \in T$ there is some $T_t \in \mathcal{N}_t$ such that*

$$\{(tx, t'x) : x \in X,\ t' \in T_t\} \subseteq U_n.$$

Conversely, given a collection $\{U_n : n \in \mathbb{N}\}$ of sets with these properties there is a T-pseudometric d on X such that

$$U_n \subseteq \left\{ (x, y) : d(x, y) < \frac{1}{2^n} \right\} \subseteq U_{n-1.}$$

If the topology on T is based at 1 then requirement (4) need only be verified at $t = 1$.

Proof. Suppose $\{U_n\}$ is a collection of sets satisfying the properties above. Let $U_0 = X \times X$, and set

$$\begin{aligned} f(x,y) &= \begin{cases} 1/2^n & \text{if } (x,y) \in U_{n-1} \setminus U_n \text{ for some } n, \\ 0 & \text{if } (x,y) \in U_n \text{ for all } n, \end{cases} \\ d(x,y) &= \bigwedge_{x\text{-}y \text{ chains}} \sum_0^n f(x_i, x_{i+1}), \end{aligned}$$

where the infimum in the definition of d is over all x-y chains, i.e., all finite sequences $x = x_0, x_1, \ldots, x_{n+1} = y$. The standard proof of the Metrization Lemma [15, 6.12] shows d to be a pseudometric with the nesting property claimed for it. It remains to show that d is continuous.

Let $d(x,y) = r$, and suppose $\epsilon > 0$ is given. Find a chain $x = x_0, x_1, \ldots, x_{n+1} = y$ such that

$$r \leq \sum_0^n f(x_i, x_{i+1}) < r + \epsilon.$$

Let $\{k_i\}$ be nonnegative integers such that $(x_i, x_{i+1}) \in U_{k_i}$ and $\sum_0^n 1/2^{k_i+1} < r + \epsilon$. (If $(x_i, x_{i+1}) \in U_{n-1} \setminus U_n$ choose k_i to be $n-1$. If $(x_i, x_{i+1}) \in \bigcap_{\mathbb{N}} U_n$ choose k_i large enough to make the inequality valid.) Now U_{k_i} is open, so there are open sets $V_i \in \mathcal{N}_{x_i}$ such that $V_i \times V_{i+1} \subseteq U_{k_i}$. We claim that $d(x', y') < r + \epsilon$ for any $(x', y') \in V_0 \times V_{n+1}$. This follows directly from the existence of the chain $x' = z_0, \ldots, z_{n+1} = y'$, where $z_i = x_i$ for $1 \leq i \leq n$. For $(z_i, z_{i+1}) \in U_{k_i}$ for $0 \leq i \leq n$, with the consequence that

$$d(x', y') \leq \sum_0^n f(z_i, z_{i+1}) \leq \sum_0^n \frac{1}{2^{k_i+1}} < r + \epsilon.$$

It remains to find a neighborhood of (x,y) within which the value of d lies above $r - \epsilon$. But this is easy. Let n be a positive integer large enough that $1/2^n < \epsilon$. Since (x,x) lies in the open set U_n there is some $V_x \in \mathcal{N}_x$ such that $V_x \times V_x \subseteq U_n$, and likewise some $V_y \in \mathcal{N}_y$ such that $V_y \times V_y \subseteq U_n$. We claim that $d(x', y') > r - \epsilon$ for all $(x', y') \in V_x \times V_y$. For $(x, x') \in U_n$ implies that $d(x, x') \leq f(x, x') \leq 1/2^{n+1}$, and likewise $d(y, y') \leq 1/2^{n+1}$. Therefore

$$r = d(x,y) \leq d(x,x') + d(x',y') + d(y',y) \leq d(x',y') + \frac{1}{2^n},$$

from which it follows that $d(x', y') \geq r - 1/2^n > r - \epsilon$. ∎

Definition . *Let X be a flow. The T-*uniformity *on X is the uniformity generated by the T-pseudometrics. The T-*topology *on X is the topology of the T-uniformity.*

Note that the T-topology is always coarser than the given topology on X because the T-pseudometrics are continuous. It may be properly coarser, however, as we see in Example 1.

Theorem 6. *For a positive bounded real-valued function g on a flow X the following are equivalent.*

1. *g is T-uniformly continuous.*
2. *For every $t \in T$ and $\epsilon > 0$ there is some $T_t \in \mathcal{N}_t$ such that for all $t' \in T_t$*
$$|gt - gt'| < \epsilon.$$
3. *For any $T\ell$-subgroup G of CX which includes g, gt lies in the uniform closure of $G(g, t)$ for every $t \in T$.*

Proof. Suppose g is T-uniformly continuous, and that $\epsilon > 0$ is given. Find a T-pseudometric d and $\delta > 0$ such that $d(x, y) < \delta$ implies $|gx - gy| < \epsilon$ for all $x, y \in X$. For a given $t \in T$ find $T_t \in \mathcal{N}_t$ such that $d(tx, t'x) < \delta$ for all $x \in X$ and $t' \in T_t$. Then clearly $|gt - gt'| < \epsilon$ for all $t' \in T_t$. Now suppose g satisfies (2). Define $d(x, y) = |gx - gy|$. Then d is a T-pseudometric, and $d(x, y) < \epsilon$ implies $|gx - gy| < \epsilon$ for all $x, y \in X$.

Let G be as in (3). If $g \in G$ satisfies (2) then it follows that $gt - \epsilon \in G(g, t)$ for all $t \in T$ and $\epsilon > 0$. But since the sequence $gt - \frac{1}{n}$ converges uniformly to gt, (3) follows. Conversely, if (3) holds then in particular it holds for $G = CX$. Then for any $t \in T$ and $\epsilon > 0$ there is some $h \in G(g, t)$ such that $|gt - h| < \epsilon$. That means that there is some $T_t \in \mathcal{N}_t$ such that $gt' \geq h \geq gt - \epsilon$ for all $t' \in T_t$. By arguing as before with $n - g$ in place of g for sufficiently large n, we get another neighborhood T_t' of t such that $gt'' \leq gt + \epsilon$ for all $t'' \in T_t'$. Therefore $|gt - gt''| < \epsilon$ for all $t'' \in T_t \cap T_t'$. That is, 2 holds. □

Theorem 6 motivates the following terminology.

Definition . *Suppose G is an abstract ℓ-group on which T acts. An element $g \in G$ will be called* T-uniformly continuous *if for all $t \in T$ and $n \in N$ there is some $T_t \in \mathcal{N}_t$ such that for all $t' \in T_t$,*
$$n\,|gt - gt'| < 1.$$
We use G^T to denote the collection of T-uniformly continuous elements of G. In particular, if $G = CX$ then we refer to these elements as T-uniformly continuous functions*, and denote the collection of them $C^T X$.*

The preceding is the central definition of this article, and deserves several comments.

- A T-uniformly continuous function on a space on which T acts must be continuous.
- To the best of our knowledge, these functions were first exploited by Auslander in [2], p. 121.
- Even in abstract ℓ-groups, we reserve the right to use the more suggestive version of T-uniform continuity in condition (2) of Theorem 6: for all $t \in T$ and $\epsilon > 0$ there is some $T_t \in \mathcal{N}_t$ such that $|gt - gt'| < \epsilon$ for all $t' \in T_t$. Strictly speaking, the latter does not make sense in an abstract ℓ-group because there need be no element corresponding to the constant function ϵ. However, by replacing ϵ by $1/n$ and multiplying the inequalities by n, one obtains the definition above. Moreover, in every instance the arguments with ϵ translate into valid ℓ-group arguments when so transformed.
- If the topology on T is based at 1 then it is enough to check the condition displayed above at $t = 1$.

We note that the idea of uniform convergence can be expressed in purely ℓ-group terms. Corr 4

Definition . *Let G be an ℓ-group, (g_n) a sequence in G and $g \in G$. Then (g_n)* converges uniformly *to g if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $|g_n - g| < \varepsilon$ whenever $n \geq N$. An ℓ-subgroup H of G is* uniformly closed *if all uniform limits of sequences in H are in H.*

Proposition 7. *In any ℓ-group G acted upon by T, G^T forms a T-invariant uniformly closed ℓ-subgroup of G .*

Proof. Suppose $0 < g_1, g_2 \in G^T$, $t \in T$, and $\epsilon > 0$ are given, and find $T_1, T_2 \in \mathcal{N}_t$ such that $|g_i t - g_i t_i| < \epsilon/2$ for all $t_i \in T_i$. Let $T_t = T_1 \cap T_2$. Then for all $t_3 \in T_t$ we have

$$\begin{aligned} |(g_1 + g_2)\, t - (g_1 + g_2)\, t_3| &\leq |g_1 t - g_1 t_3| + |g_2 t - g_2 t_3| < \epsilon, \\ |(g_1 \vee g_2)\, t - (g_1 \vee g_2)\, t_3| &\leq |g_1 t - g_1 t_3| \vee |g_2 t - g_2 t_3| < \epsilon. \end{aligned}$$

This shows the closure of G^T under sum and supremum, and since it is clearly closed under negation and contains 1, it is an ℓ-subgroup of G. To show that G^T is T-invariant consider $0 < g \in G^T$ and $t_2 \in T$; to test the membership of gt_2 in G^T consider $t_1 \in T$ and $\epsilon > 0$. First find $T_3 \in \mathcal{N}_{t_2 t_1}$ such that $|gt_2 t_1 - gt_3| < \epsilon$ for all $t_3 \in T_3$. Then use the continuity of multiplication in T to find $T_i \in \mathcal{N}_{t_i}$ such that $T_2 T_1 \subseteq T_3$. Thus for all $t_1' \in T_1$ we have $|gt_2 t_1 - gt_2 t_1'| < \epsilon$, i.e., $gt_2 \in G^T$. Finally consider a function $h \in G$ in the uniform closure of G^T, and consider given $t \in T$ and $\epsilon > 0$. Then there is some $g \in G^T$ such that $|h - g| < \epsilon/3$, and for g there is some $T_1 \in \mathcal{N}_t$ such that

$|gt - gt_1| < \epsilon/3$ for all $t_1 \in T_1$. Then for all $t_1 \in T_1$ we have

$$|ht - ht_1| \leq |ht - gt| + |gt - gt_1| + |gt_1 - ht_1| < \epsilon. \blacksquare$$

5. THE YOSIDA ADJUNCTION FOR FLOWS

In this section we develop the Yosida adjunction for flows. In so doing we recapture the classical Yosida representation [20], which is the case in which $T = 1$. We carry out this development from first principles, in part because it is no more difficult than first outlining the classical representation and then extending it to flows. But a more basic reason for this development is that we can simplify the proofs considerably, since we can identify the points of the Yosida space of an ℓ-group G with the ℓ-homomorphisms $x : G \to \mathbb{R}$, as a consequence of the fact that we are interested only in bounded real-valued functions in this article. Corr 5

Definition . *A* $T\ell$-group *is an ℓ-group G acted upon by T so that $G^T = G$. A* $T\ell$-homomorphism *is an ℓ-homomorphism f between $T\ell$-groups which commutes with the actions, i.e., such that $ft = tf$ for all $t \in T$. The category of $T\ell$-groups with $T\ell$-homomorphisms is denoted* **TlGp**.

We show in Theorem 20 that the canonical examples of $T\ell$-groups are those of the form CX for compact flows X.

Proposition 8. *Suppose G and H are ℓ-groups on which T acts, and that f commutes with the actions. Then f takes G^T into H^T; that is, f "drops" to make this diagram commute.*

$$\begin{tikzcd}
G \arrow[r, "f"] & H \\
G^T \arrow[u] \arrow[r, "f^T"'] & H^T \arrow[u]
\end{tikzcd}$$

Proof. Given $g \in G^T$ set $h = fg$. For $t \in T$ and $\epsilon > 0$ find a positive integer n such that $1/n < \epsilon$, then find $T_t \in \mathcal{N}_t$ such that $n\,|gt - gt'| < 1$ for all $t' \in T_t$. Since f takes the strong unit of G to that of H and commutes with the actions, we get $n\,|ht - ht'| \leq 1$ for all $t' \in T_t$. From this follows $|ht - ht'| < \epsilon$ for all $t' \in T_t$, i.e., $h \in H^T$. □ Corr 6

Corollary 9. *The class of $T\ell$-groups is closed under products (on which T acts componentwise), subobjects (i.e., $T\ell$-subgroups), and $T\ell$-homomorphic images.*

For a continuous function $f : X \to Y$ we use the notation $Cf : CY \to CX$ for the ℓ-homomorphism defined by the rule $(Cf)\, g = gf$ for $g \in CY$. In particular, if T acts on a space X then there is a corresponding action of T on CX; we consider no other action on CX in this article. However, instead of denoting the action on CX corresponding to $t \in T$ by Ct, we use the simpler notation t instead.

Proposition 10. *If X and Y are spaces on which T acts and if f commutes with the actions, then Cf also commutes with the actions.*

Proof. The commutativity of the square on the left is a direct consequence of the commutativity of the square on the right. □

$$\begin{tikzcd}
CX & CX \arrow[l, "t"'] & X \arrow[r, "t"] \arrow[dd, "f"'] & X \arrow[dd, "f"] \arrow[rd, "hf"] & \\
 & & & & R \\
CY \arrow[uu, "Cf"] & CY \arrow[l, "t"] \arrow[uu, "Cf"'] & Y \arrow[r, "t"'] & Y \arrow[ru, "h"'] &
\end{tikzcd}$$

Definition . *We use $C^T f$ to denotes the restriction of Cf to $C^T Y$.*

Observe that it is Proposition 8 which assures that $C^T f$ carries $C^T Y$ into $C^T X$.

Corollary 11. *C^T is a functor from the category* **TSp** *of flows to the category* **TlGp** *of $T\ell$-groups.*

The *Yosida space* YG of an ℓ-group G is the set $\hom_{\mathbf{lGp}}(G, R)$ of ℓ-homomorphisms from G into the ℓ-group $\mathbb{R}$ of real numbers. The topology on YG has a base consisting of sets of the form

$$\underset{YG}{\operatorname{coz}}\, \bar{g} = \{y : yg \neq 0\}, \quad g \in G.$$

The notation $\operatorname{coz} \bar{g}$ for the basic open set associated with g is justified by Proposition 14. Until then, however, we treat $\operatorname{coz} \bar{g}$ as simply the name of this set.

Lemma 12. *Let G be an ℓ-group. Then the following identities hold in YG for all $g_i \in G^+$.*

$$\operatorname{coz} \overline{g_1 \vee g_2} = \operatorname{coz} \bar{g}_1 \cup \operatorname{coz} \bar{g}_2$$
$$\operatorname{coz} \overline{g_1 \wedge g_2} = \operatorname{coz} \bar{g}_1 \cap \operatorname{coz} \bar{g}_2$$

Furthermore, $\operatorname{coz} \bar{g} = \emptyset$ *if and only if $g = 0$ in G.*

Proof. The displayed identities follow directly from the fact that each $y \in YG$ is an ℓ-homomorphism. Consider now some $0 < g \in G$. If $ng \geq 1$ for some positive integer n then choose an ℓ-ideal K maximal with respect to omitting 1. (Such ideals must exist by Zorn's Lemma, since 0 is an ℓ-ideal omitting 1 and since the family of such ℓ-ideals is closed under unions of chains.) Then K must also omit g, and so the ℓ-homomorphism $y \in YG$ with kernel K maps g to a positive real number, and so lies in $\operatorname{coz} \bar{g}$.

Now suppose no such positive integer n exists. By exchanging g for a positive multiple, we may assume by the archimedean property that Corr 7 $g \nleq 1$, i.e., that $(g-1)^{+} > 0$. Therefore the ℓ-ideal generated by $(g-1)^{-}$ is proper, since it is contained in the ℓ-ideal

$$\left\{h \in G : |h| \wedge (g-1)^{+} = 0\right\},$$

which is itself proper because it omits $(g-1)^{+}$. Find as before an ℓ-ideal K maximal among those containing $(g-1)^{-}$ and omitting 1, and let $y \in YG$ be the ℓ-homomorphism with kernel K.

We claim that $2((g-1)^{-} \vee g) \geq 1$. To establish this claim it is enough to show that $2((1-g) \vee g) \geq 1$ because

$$(g-1)^{-} = (1-g)^{+} = (1-g) \vee 0.$$

But this is simply the application of the ℓ-group identity $2\,(a \vee b) \geq a + b$ to $a = 1 - g$ and $b = g$. (The identity itself is easy to verify: $a \vee b \geq a$ and $a \vee b \geq b$ imply $2\,(a \vee b) \geq a + b$.) From the claim we deduce that K cannot contain g, and hence that y assigns a positive real number to g, i.e., that $y \in \operatorname{coz} \bar{g}$. □

Proposition 13. *For any ℓ-group G, YG is a nonempty compact Hausdorff space.*

Proof. Set $Y = YG$. Then Y is nonempty because $\operatorname{coz} \bar{1}$ is nonempty by Lemma 12. If $y_1 \neq y_2$ in Y there must be some $g \in G$ such that, say, $y_1 g < y_2 g$. Let p/q be a rational number such that $y_1 g < p/q < y_2 g$, i.e.,

$$y_1\,(qg) = q y_1 g < p < q y_2 g = y_2\,(qg)\,.$$

Set $h = p - qg \in G$, $g_1 = h^{+}$, and $g_2 = h^{-}$. Then g_1 and g_2 are disjoint strictly positive elements whose cozero sets are therefore disjoint. Since $y_i \in \operatorname{coz} \bar{g}_i$, Y is Hausdorff.

Suppose we are given an open cover of Y; without loss of generality we may assume it to be of the form $\{\operatorname{coz} \bar{g}_0 : g_0 \in G_0\}$ for some $G_0 \subseteq G^{+}$. We claim that the ℓ-ideal L generated by G_0 must contain 1. For if not then L is contained in an ℓ-ideal K maximal among those containing L but omitting 1. Let y be the ℓ-homomorphism from G into $\mathbb{R}$ with kernel K. Then we have $y g_0 = 0$ for all $g_0 \in G_0$, i.e., Corr 8 $y \notin \bigcup \operatorname{coz} \bar{g}_0$, contrary to assumption. This establishes the claim, which shows that there is some finite subset $G_1 \subseteq G_0$ such that the ideal generated by G_1 contains 1. From this it follows that $\{\operatorname{coz}\bar{g}_1 : g_1 \in G_1\}$ covers Y. For if $y \notin \bigcup_{G_1} \operatorname{coz}\bar{g}_1$ then $yg_1 = 0$ for all $g_1 \in G_1$, i.e., G_1 is contained in the kernel of y. Since the kernel of y is an ℓ-ideal omitting 1, this last state of affairs violates the assumption that the ℓ-ideal generated by G_1 contains 1, and completes the proof. □

For an element g in an abstract ℓ-group G we use $\bar{g}$ to denote the real-valued function on YG defined by the rule $\bar{g}y = yg$.

Proposition 14. *For any element g of an ℓ-group G, $\bar{g}$ is a continuous function such that*

$$\operatorname{coz}\bar{g} = \{y \in YG : \bar{g}y \neq 0\}.$$

Proof. Given $g \in G$, $y \in Y = YG$, and $\epsilon > 0$, find rational numbers p/q and s/t such that

$$\bar{g}y - \epsilon < \frac{p}{q} < \bar{g}y < \frac{s}{t} < \bar{g}y + \epsilon,$$

i.e., $pt < qt\bar{g}y < sq$. Let $h = (sq - qtg)^{+} \wedge (qtg - pt)^{+}$, so that $y' \in \operatorname{coz} h$ if and only if $pt < qt\bar{g}y' < sq$. Thus $\operatorname{coz} h$ is an open set containing y such that $|gy' - gy| < \epsilon$ for all $y' \in \operatorname{coz} h$. This verifies that $\bar{g}$ is continuous. □

The *Yosida representation of the ℓ-group* G is the mapping $\mu_G : G \to CYG$ defined by the rule $\mu_G g = \bar{g}$ for all $g \in G$.

Proposition 15. *For any ℓ-group G, μ_G is an ℓ-injection.*

Proof. For $g_i \in G$ and $y \in YG$ we have

$$\begin{aligned} \bar{g}_1 y + \bar{g}_2 y &= yg_1 + yg_2 = y\,(g_1 + g_2) = \overline{(g_1 + g_2)}y, \text{ and} \\ \bar{g}_1 y \vee \bar{g}_2 y &= yg_1 \vee yg_2 = y\,(g_1 \vee g_2) = \overline{(g_1 \vee g_2)}y. \end{aligned}$$

And μ_G is injective because $g > 0$ implies $\operatorname{coz}\bar{g} \neq \emptyset$ by Lemma 12. □

The preceding proposition shows that any abstract ℓ-group is (ℓ-isomorphic to) an ℓ-subgroup of CX for a compact Hausdorff space X. This observation permits the reader to verify computations in ℓ-groups by carrying them out in CX.

The functor Y applies to ℓ-homomorphisms as well as to ℓ-groups. For an ℓ-homomorphism $k : G \to H$ let $Yk : YH \to YG$ be the function given by the rule $(Yk)\,y = yk$.

Proposition 16. *The map $Yk : YH \to YG$ is continuous for any ℓ-homomorphism $k : G \to H$.*

$$\begin{tikzcd}
 & G \arrow[ld, "yk"'] \arrow[dd, "k"] & & YG \\
R & & & \\
 & H \arrow[lu, "y"] & & YH \arrow[uu, "Yk"']
\end{tikzcd}$$

Proof. We need only verify that $(Yk)^{-1}(\operatorname{coz}_{YG}\bar{g}) = \operatorname{coz}_{YH}\overline{kg}$.

$$\begin{aligned}
y \in (Yk)^{-1}\left(\operatorname*{coz}_{YG}\bar{g}\right) \iff & \quad (Yk)\,y \in \operatorname*{coz}_{YG}\bar{g} \iff yk \in \operatorname{coz}\bar{g} \iff \\
\bar{g}\,(yk) \neq 0 \iff & \quad (yk)\,g \neq 0 \iff y\,(kg) \neq 0 \iff \\
\overline{kg}y \neq 0 \iff & \quad y \in \operatorname*{coz}_{YH}\overline{kg}. \ \blacksquare
\end{aligned}$$

This observation is the first step in adding actions to the Yosida representation. Suppose now that T acts on the ℓ-group G. Then each action t on G gives rise to the corresponding action Yt on YG, so that T also acts on YG. Such actions are the only ones we consider on YG, though we simplify the notation by writing Yt as t.

Definition . *For a $T\ell$-group G we let Y^TG denote the space YG acted upon by T.*

Note that the difference between YG and Y^TG lies not in the object itself, since the underlying spaces are the same, but in the additional structure on Y^TG provided by the actions.

Suppose now that G is an ℓ-group on which T acts. As we have already mentioned above, each action t on G induces the action Yt on YG, which in turn induces the action CYt on CYG. The natural question is whether the Yosida representation of G respects the actions. The answer is that it does.

Proposition 17. *For an ℓ-group G on which T acts, μ_G commutes with the actions.*

Proof. For $t \in T$, $g \in G$, and $y \in Y^TG$,

$$(t\bar{g})\,y = \bar{g}\,(ty) = (ty)\,g = y\,(tg) = \overline{tg}y. \ \blacksquare$$

Definition . *For a Tl-group G we let μ_G^T denote μ_G, considered as a $T\ell$-morphism from G into C^TY^TG.*

Proposition 18. *For any $T\ell$-group G, Y^TG is a compact flow.*

Proof. By Propositions 8 and 17, $\bar{G} = \{\bar{g} : g \in G\}$ is a $T\ell$-subgroup of C^TY^TG which separates the points of the compact space Y^TG, and thus determines its topology. Because $(gt - \epsilon)^+ \in G\,(g,t)$ for all $g \in G^+$, $t \in T$, and $\epsilon > 0$, $\bar{G}$ clearly satisfies condition (3) of Theorem 2. Therefore evaluation is continuous on Y^TG. $\square$

Proposition 19. *For any compact flow X, CX is a $T\ell$-group.*

Proof. Let $G = CX$, and consider $g \in G^+$, $t \in T$, and $\epsilon > 0$. Then the conditions of Theorem 2 hold because G determines the topology on X. Therefore for each element x in the closure D of $\operatorname{coz}(gt - \epsilon)^+$ there is some $h_x \in G(g,t)$ such that $h_x x > (gt - \epsilon)^+ x$. Since

$$\left\{ \operatorname{coz}\left(h_x - (gt - \epsilon)^+\right)^+ : x \in D \right\}$$

covers the compact set D, and since $G(g,t)$ is closed under finite suprema, there is a single $h \in G(g,t)$ such that $hx > (gt - \epsilon)$ for all $x \in D$, i.e., $h \geq (gt - \epsilon)^+$. This gives a neighborhood T_1 of t such that $gt_1 \geq h \geq gt - \epsilon$ for all $t_1 \in T_1$. By arguing similarly with $n - g$ in place of g, where n is a positive integer such that $gx < n$ for all $x \in X$, we get a second neighborhood T_2 of t such that

$$n - gt_2 = (n - g)\, t_2 \geq (n - g)\, t - \epsilon = n - gt - \epsilon$$

for all $t_2 \in T_2$. But then we get $|gt - gt_3| < \epsilon$ for all $t_3 \in T_1 \cap T_2$, i.e., g is T-uniformly continuous. □

Theorem 20. *The following are equivalent for an ℓ-group G on which T acts.*

1. *G is a $T\ell$-group.*
2. *G is $T\ell$-isomorphic to a $T\ell$-subgroup of $C^T X$ for some flow X.*
3. *G is $T\ell$-isomorphic to a $T\ell$-subgroup of CX for some compact flow X.*
4. *There is some compact flow X such that G is $T\ell$-isomorphic to a $T\ell$-subgroup of CX which separates the points of X.*
5. *For all $g \in G^+$ and $t \in T$, gt lies in the uniform closure of $G(g,t)$.*

If the topology on T is based at 1 it is enough to verify condition (5) at $t = 1$.

Proof. The implication from (1) to (4) is provided by μ_G^T, (3) follows clearly from (4), (2) follows from (3) by Proposition 19, and (1) follows from (2) by virtue of the fact that an ℓ-subgroup of a $T\ell$-group is itself a $T\ell$-group. Finally, the equivalence of (1) and (5) is part of Theorem 6. □

The next result shows that, in the terminology of [13, 26.1], $\left(\mu_G^T, Y^T G\right)$ is a C^T-universal map for G. The reader should keep in mind that C^T is a contravariant functor.

Theorem 21. *For any $T\ell$-group G there is a compact flow $Y^T G$ and a $T\ell$-embedding μ_G^T which is universal in the following sense. Given any*

Corr 9 *space X on which T acts and any $T\ell$-morphism k there is a unique flow map f such that the diagram commutes. The flow map f is injective*

$$\begin{tikzcd}
G \arrow[r, "\mu_G^T"] \arrow[dr, "k"'] & C^T Y^T G \arrow[d, "C^T f"] & Y^T G \\
& C^T X & X \arrow[u, "f"']
\end{tikzcd}$$

if and only if kG separates the points of X, and f is a bicontinuous injection, i.e., the insertion of a subspace, if and only if kG separates the points and determines the topology on X. Finally, k is injective if and only if fX is dense in $Y^T G$.

Proof. The requirement that the diagram commutes means that for all $g \in G$ and $x \in X$,

$$(fx)\,g = \bar{g}\,(fx) = \left(\left(C^T f\right)\bar{g}\right)x = (kg)\,x.$$

Accordingly for $x \in X$ define fx by this rule. Then it is easy to verify that fx is an ℓ-homomorphism from G into R, which is to say an element of $Y^T G$. Furthermore, the function f is continuous because $f^{-1}\left(\operatorname{coz}_{Y^T G}\bar{g}\right)$ is

$$\{x : \bar{g}\,(fx) \neq 0\} = \{x : (fx)\,g \neq 0\} = \{x : (kg)\,x \neq 0\},$$

which is $\operatorname{coz}_X(kg)$, and f commutes with the actions because

$$\begin{aligned}
((ft)\,x)\,g &= (f\,(tx))\,g = (kg)\,(tx) = (t\,(kg))\,x = ((tk)\,g)\,x = \\
((kt)\,g)\,x &= (k\,(tg))\,x = (fx)\,(tg) = (t\,(fx))\,g = ((tf)\,x)\,g.
\end{aligned}$$

Now kG separates the points of X if and only if for all $x_1 \neq x_2$ in X there is some $g \in G$ for which

$$(fx_1)\,g = (kg)\,x_1 \neq (kg)\,x_2 = (fx_2)\,g,$$

i.e., $fx_1 \neq fx_2$. And kG determines the topology on X if and only if for every open set $U \subseteq X$ and element $x \in U$ there is some $0 < g \in G$ for which $x \in \operatorname{coz}(kg) \subseteq U$. But considering that $\operatorname{coz}(kg) = f^{-1}(\operatorname{coz}\bar{g})$, the latter condition is equivalent to the bicontinuity of f.

Finally suppose that fX is dense in $Y^T G$ and consider $0 < g \in G$. Then there is some $y \in Y^T G$ such that $\bar{g}y > 0$, and since fX is dense in $Y^T G$ we may assume $y \in fX$, say $fx = y$ for some $x \in X$. Then

$$(kg)\,x = (fx)\,g = yg = \bar{g}y > 0,$$

meaning $kg > 0$ in $C^T X$. Conversely assume that fX is not dense in $Y^T G$, and find $0 < g \in G$ for which $\operatorname{coz} g$ is disjoint from fX. Then for all $x \in X$

$$(kg)\,x = (fx)\,g = \bar{g}\,(fx) = 0,$$

which is to say that k is not injective. $\square$

Definition . *For a $T\ell$-homomorphism $k : G \to H$ let $Y^T k$ denote the flow map f produced from $\mu_H^T k$ as in Theorem 21.*

Both $Y^T$and C^T are contravariant functors, and both become covariant when either **TSp** or **TlGp** (but not both) is replaced by its opposite. If **TSp** is replaced by its opposite then Y^T becomes left adjoint to C^T, and if **TlGp** is replaced by its opposite then C^T becomes left adjoint to Y^T [13, 27.6]. (The category $\mathbf{TSp}^{\mathrm{op}}$ has some interest, since it is equivalent to the category of compact regular frames acted upon by T. This is the subject of [6]. We know of no reason to be interested in $\mathbf{TlGp}^{\mathrm{op}}$.) We summarize the situation as follows.

Corollary 22. *C^T and Y^T are adjoints of one another.* Corr 10

6. Compactification of flows

Definition . *For a flow X let $\beta^T X$ denote the compact flow $Y^T C^T X$, and let $\beta_X^T : X \to \beta^T X$ denote the flow map produced by taking k to be the identity map on $G = C^T X$ in Theorem 21.*

As a direct consequence of Theorem 21 (see [13]) we have the flow counterpart of the Stone-Čech compactification of a space. Our next result shows that the Stone-Čech compactification possesses the expected universal properties. (For another construction of the Stone-Čech compactification, see Remark 1 at the end of this section.)

Theorem 23. *β^T is a functor. That is, any flow map f lifts uniquely to a flow map $\beta^T f$ which makes this diagram commute.*

$$\begin{tikzcd}
\beta^T X \arrow[r, "\beta^T f"] & \beta^T Y \\
X \arrow[u, "\beta_X^T"] \arrow[r, "f"'] & Y \arrow[u, "\beta_Y^T"']
\end{tikzcd}$$

In particular, any flow map f from X into a compact flow Y lifts uniquely to a flow map $\beta^T f$ from $\beta^T X$ into Y such that $\beta^T f \beta_X^T = f$. Corr 11

Definition . *A flow X is* compactifiable *if it is (flow homeomorphic to) a subflow of a compact flow.*

Theorem 24. *The following are equivalent for a flow X.*

(1) *X is compactifiable.*

(2) *β_X^T is a bicontinuous injection, i.e., a subspace insertion.*
(3) *$C^T X$ separates the points and determines the topology on X.*
(4) *For every closed subset $Y \subseteq X$ and every $x \notin Y$ there is a T-pseudometric d on X such that $d(x,y) \geq 1$ for all $y \in Y$.*
(5) *The T-topology is the given topology on X.*

Proof. The equivalence of (2) and (3) is an application of the penultimate assertion of Theorem 21, and (2) clearly implies (1). To establish the implication from (1) to (2), suppose that X is a subflow of a compact flow Y. Then all the real-valued functions on Y are T-uniformly continuous, and their restrictions to X therefore constitute a $T\ell$-subgroup of $C^T X$ which separates the points and determines the topology on X. To verify the implication from (3) to (4), observe that for any closed $Y \subseteq X$ and $x \notin Y$ there is some $g \in C^T X$ such that $gx = 0$ and $gy = 1$ for all $y \in Y$. The desired T-pseudometric is then

$$d(x', y') = |gx' - gy'| .$$

The equivalence of (4) and (5) is clear. Finally, if (4) holds then set $gy = 0 \vee (1 \wedge d(x,y))$ to get a demonstrably bounded T-uniformly continuous function which separates x from Y. This shows that (3) must hold if (4) does. □

Corollary 25. *Every flow X has a finest compactifiable quotient flow $Y = X/\sim$, where $x \sim y$ if and only if $gx = gy$ for all $g \in C^T X$, and $\beta^T X = \beta^T Y$.*

Corollary 26. *Any topological group, considered as a flow acting on itself by left multiplication, is compactifiable.*

Proof. We argue by means of part (4) of Theorem 24. Consider a closed subset $S \subseteq T$ and element $t_0 \notin S$. Let T_1 be a neighborhood of 1 small enough that $T_1 t_0 \cap S = \emptyset$; having chosen $T_n \in \mathcal{N}_1$, find $T_{n+1} \in \mathcal{N}_1$ such that $T_{n+1} = T_{n+1}^{-1}$ and

$$T_{n+1} \cdot T_{n+1} \cdot T_{n+1} \subseteq T_n.$$

If we set

$$U_n = \operatorname{int} \{(t, t_n t) : t \in T,\, t_n \in T_n\} ,$$

then it is routine to verify that $\{U_n\}$ satisfies the conditions of Theorem 5, and hence produce a T-pseudometric d such that

$$\left\{(s_1, s_2) : d(s_1, s_2) < \frac{1}{4}\right\} \subseteq U_1.$$

Therefore $4d(t_0, s) \geq 1$ for all $s \in S$. □

Corollary 27. *Suppose T acts on a compact space X. Then X is a flow if and only if for every neighborhood U of the diagonal and every $t \in T$ there is some $T_t \in \mathcal{N}_t$ such that*

$$\{(tx, t'x) : x \in X, t' \in T_t\} \subseteq U.$$

Proof. If X is a flow then the T-topology agrees with the given topology on X, hence the T-uniformity is the unique uniformity on X, whose entourages are all neighborhoods of the diagonal. The condition then follows from (4) of Theorem 5. On the other hand, the condition allows for the construction of T-pseudometrics by Theorem 5 in sufficient quantity to satisfy Theorem 24. □

Remark 1. *Another construction of the Stone-Čech compactification of a flow.*

We can also construct the Stone-Čech compactification of a flow by
mimicking a familiar construction from general topology. We sketch Corr 12
the details that our two approaches yield flow homeomorphic results.
Recall that $C^T X$ denotes the set of bounded T-uniformly continuous
real-valued functions on the flow X. Consider the set

$$\mathcal{F} = \{f \in C^T X : 0 \leq f(x) \leq 1 \text{ for all } x \in X\}.$$

and the map $i : X \to [0,1]^{\mathcal{F}}$ defined by $(ix)_f = f(x)$ for $f \in \mathcal{F}$. Define the Stone-Čech compactification of the flow X by setting $\beta^T X = \mathrm{cl}\,(iX)$, where cl denotes closure in $[0,1]^{\mathcal{F}}$. Also define the map $\beta^T_X : X \to \beta^T X$ in the obvious way, namely as the codomain restriction of i.

To describe the action of T on $\beta^T X$, we recall first Proposition 7, which has the following consequences.

- If $f \in \mathcal{F}$ and $t \in T$, then the function $ft \in \mathcal{F}$, where $ft(x) = f(tx)$.
- $f\,(t't) = ft'\,(t)$ for $f \in \mathcal{F}$ and $t', t \in T$.

For $t \in T$, define the action of t on $[0,1]^{\mathcal{F}}$ by $(t\omega)_f = \omega_{ft}$. By the observation above, this map is well defined. It extends the action of t on X since

$$(t(ix))_f = (ix)_{ft} = (ft)\,x = f(tx) = (i\,(tx))_f\,,$$

whence $t(ix) = i\,(tx)$. To see that T acts on $[0,1]^{\mathcal{F}}$, observe that

$$((t't)\,\omega)_f = \omega_{f(t't)} = \omega_{(ft')t} = (t\omega)_{ft'} = (t'\,(t\omega))_f\ .$$

Finally we show that $\beta^T X$ is a flow. To accomplish this, first, let
$f \in C^T X$, $t \in T$, and $\varepsilon > 0$. Then there is a neighborhood T_1 of t such
that, for all $x \in X$ and $t' \in T_1$, $|ft'x - ftx| < \varepsilon/2$. In particular, if Corr 13 and 14

$\omega \in \beta^T X$, $|t'\omega_f - t\omega_f| < \varepsilon$. Now suppose that $\omega \in \beta^T X$, and $t \in T$. For a finite $F \subseteq \mathcal{F}$ and $\varepsilon > 0$ consider the neighborhood of $t\omega$ in $\beta^T X$ given by

$$O = \left\{\omega' : \left|\omega'_f - t\omega_f\right| < \varepsilon \text{ for } f \in F\right\}.$$

Since $\mathcal{F} \subseteq C^T X$, there is a neighborhood T_1 of t such that, for all $x \in X$, $t' \in T_1$, and $f \in F$, $|ft'x - ftx| < \varepsilon/2$. In particular, if $\omega \in \beta^T X$, $|t'\omega_f - t\omega_f| < \varepsilon/2$. Define the neighborhood U of ω by

$$U = \left\{\omega' : \left|\omega'_{ft} - \omega_{ft}\right| < \varepsilon/2 \text{ for } f \in F\right\}.$$

Fix $t' \in T_1$, $\omega' \in U$, and $f \in F$. Then the easy estimate

$$\left|t'\omega'_f - t\omega_f\right| \leq \left|t'\omega'_f - t\omega'_f\right| + \left|t\omega'_f - t\omega_f\right| < \varepsilon$$

shows that $t'\omega' \in O$ and hence that $\beta^T X$ is a flow.

The fact that Theorem 23 can be proven in this setting is routine and we omit it.

7. Examples

Here is where the theory of flow compactifications diverges from the theory of space compactifications: not every Tychonoff flow is compactifiable. We give two examples. Example 1 is a discrete flow X and a monoid T such that neither the flow X nor the monoid T (acting on itself by left multiplication) is compactifiable. Example 2 is more telling; it is a non-compactifiable separable metric flow where the action monoid is a topological group. We begin with a simple lemma.

Lemma 28. *Suppose that X is a flow with the following property: for every pair of nonempty open sets U, V in X and for every neighborhood T_1 of the identity in T there exist $x \in X$ and $s, t \in T_1$ such that $sx \in U$ and $tx \in V$. Then the only T-uniformly continuous functions on X are the constants.*

Proof. Suppose that $f : X \to \mathbb{R}$ satisfies $fx_0 = 0$ and $fx_1 = 1$ for points $x_0, x_1 \in X$. Let $U = f^{-1}(4/5, \infty)$ and $V = f^{-1}(-\infty, 1/5)$. If f is T-uniformly continuous then there is a neighborhood T_1 of the identity in T such that $|fx - ftx| < 1/5$ for all $x \in X$ and $t \in T_1$. Let x, s and t be as in the statement of the lemma. Then $fsx > 4/5$ and $ftx < 1/5$, so we get the contradiction

$$\frac{3}{5} < |fsx - ftx| \leq |fsx - fx| + |fx - ftx| < \frac{2}{5}. \blacksquare$$

Example 1. *A noncompactifiable flow and a noncompactifiable monoid.*

Let $\mathbb{N}$ be the discrete space of natural numbers, acted upon by

$$T=\left\{t\in\mathbb{N}^{\mathbb{N}}:\exists\, m\quad\forall\, n\geq m\quad tn=n\right\}.$$

Topologize T by neighborhoods of $t\in T$ of the form

$$T\left(t,m\right)=\left\{t'\in T:\forall\, n\leq m\quad t'n=tn\right\}.$$

$\mathbb{N}$ is a flow, i.e., the evaluation map is continuous, but $\mathbb{N}$ cannot be embedded in a compact flow because the only T-uniformly continuous functions are the constants. To see this, consider nonempty neighborhoods U and V in $\mathbb{N}$. Suppose that $n_1\in U$ and $n_2\in V$. Then any neighborhood $T\left(1,m\right)$ contains actions t_i such that $t_i\left(m+1\right)=n_i$ for $i=1,2$. This shows that the conditions of Lemma 28 are satisfied, so no nonconstant function on $\mathbb{N}$ is T-uniformly continuous. Thus, $\mathbb{N}$ has no flow compactification. (Note that the topology on T is based at 1.)

We now turn our attention to the monoid T and show that the only T-uniformly continuous functions on T are the constants. To this end, let U_i, $i=1,2$, be nonempty open sets in T and T_1 a neighborhood of the identity in T. Find an integer m and $t_1,t_2\in T$ such that $U_i\supseteq T\left(t_i,m\right)$ and $T_1\supseteq T\left(1,m\right)$. Define $s\in T$ by

$$s(j)=\begin{cases} m+j & \text{if}\quad 1\leq j\leq m\\ j & \text{if}\quad j>m\end{cases}$$

and $s_i\in T_1$ for $i=1,2$ by

$$s_i(j)=\begin{cases} j & \text{if}\quad j\leq m \text{ or } j>2m\\ t_i\left(j-m\right) & \text{if}\quad m+1\leq j\leq 2m.\end{cases}$$

Then for $1\leq j\leq m$,

$$s_is(j)=s_i(m+j)=t_i(j)$$

which shows that $s_is\in U_i$. □

As a final remark, we observe that the actions in T on $\mathbb{N}$ can be extended to $\beta\mathbb{N}$ by requiring that $t\omega=\omega$ for all $\omega\in\beta\mathbb{N}\diagdown\mathbb{N}$. So T acts on $\beta\mathbb{N}$ and $\mathbb{N}$ is dense and C^*-embedded, yet $\beta\mathbb{N}$ is not a flow. An example with these properties was promised following the proof of Corollary 4.

Example 2. *A noncompactifiable flow with a topological group of actions.*

It is reasonable to expect that a Tychonoff flow X acted upon by a Hausdorff topological group would be compactifiable. The argument might go along the lines of the proof of Corollary 26 as follows. For each neighborhood $T_1\in\mathcal{N}_1$ define the entourage

$$U\left(T_1\right)=\left\{\left(x,t_1x\right):x\in X,\, t_1\in T_1\right\}.$$

Then for given $x \neq y$ in X it is easy to pick out a sequence $\{U_n\}$ such that $(x,y) \notin U_1$ and such that $\{U_n\}$ satisfies requirements (2), (3), and (4) of Theorem 5. The only obstacle is that these entourages need not be open. Put another way, we can produce a pseudometric to separate x from y which is uniformly continuous with respect to the uniformity with the displayed entourages, and the pseudometric will even respect the actions in the sense of the definition of T-pseudometric, but we cannot guarantee that this pseudometric will be continuous as a function from $X \times X$ into $\mathbb{R}$.

We give an example of a topological group T acting on a separable metrizable flow $(X, \mathcal{T}_1)$ which nevertheless is not compactifiable. This space has the property that T-uniformly continuous functions on X separate the points of X, but the topology $\mathcal{T}_1$ on X fails to be generated by cozero sets of T-uniformly continuous functions. However, we show that there is a topology $\mathcal{T}_2$ on X, weaker than $\mathcal{T}_1$, so that the space $(X, \mathcal{T}_2)$ is a compact metric space which is still a flow under the actions of the group T. The topology $\mathcal{T}_2$ arises precisely as described in Corollary 25; that is, $\mathcal{T}_2$ **is** generated by cozero sets of T-uniformly continuous functions on X.

Let X be formed from a disjoint union of a countable number of copies $I_1, I_2, \cdots$ of the unit interval $I = [0,1]$ with the 0's identified. To obtain a concrete realization of X, let ℓ_2 denote the real Hilbert space of square summable sequences and

$$X = \{(x_n) \in \ell_2 : x_n \neq 0 \text{ for at most one } n, \text{ and } x_n \in I \text{ for all } n\}.$$

Put $X_n = X \cap I_n$ and identify X_n with I. We assume without further mention the usual topological properties on $X_n \setminus \{0\}$ and order properties on X_n induced by the obvious isometry with I.

The topology $\mathcal{T}_1$ is the restriction of the norm topology on ℓ_2 to X, and the topology $\mathcal{T}_2$ is the restriction of the weak topology on ℓ_2 to X. It is clear that a neighborhood base for $x \neq 0$ (in each topology) is generated by open intervals in X_n. We now describe a base of neighborhoods of $0 \in X$ in each topology. In $\mathcal{T}_1$, the sets

$$V_\varepsilon = \{x \in X : ||x|| < \varepsilon\}, \qquad \varepsilon > 0,$$

form a base of neighborhoods of 0, while in $\mathcal{T}_2$, the base consists of the sets

$$W(F, \varepsilon) = \{x \in X : 0 \leq x < \varepsilon \text{ for } x \in X_n,\, n \in F\},$$

where F is a finite subset of $\mathbb{N}$ and $\varepsilon > 0$. Obviously the space $(X, \mathcal{T}_1)$ is a separable metric space, $(X, \mathcal{T}_2)$ is compact metric space, and the identity map is a continuous bijection from $(X, \mathcal{T}_1)$ onto $(X, \mathcal{T}_2)$. The topology $\mathcal{T}_1$ fails to be locally compact only because no neighborhood of 0 is compact.

Define T to be the set of those actions t with the following properties.

- t is a homeomorphism with respect to $\mathcal{T}_1$.
- t maps X_n onto X_n; let t_n denote $t|_{X_n}$.
- t_n is the identity map on X_n for all but finitely many n.
- There is some $\varepsilon > 0$ such that t is the identity on V_ε.

Clearly T is a group, and its actions are homeomorphisms with respect to $\mathcal{T}_2$.

To describe the topology on T, we distinguish two distinct types of neighborhoods. First, let

$$T_n = \{t \in T : t_k (1/n) = 1/n \text{ for all } k\}.$$

We note that $t \in T_n \Longleftrightarrow t(U_n) = U_n$ where $U_n = V_{1/n}$ for $n = 1, 2, \cdots$. Second, if O_1 and O_2 are open intervals of X_n, define

$$S(O_1, O_2) = \{t : t(O_1) = O_2\}.$$

A subbase of neighborhoods in T consists of sets of the form $t_1 T_n t_2$ or of the form $S(O_1, O_2)$.

The following verifications are routine.

Lemma 29. *T is a topological group and both $(X, \mathcal{T}_1)$ and $(X, \mathcal{T}_2)$ are flows under the action of T.*

Proof. We check only subbasic open sets. The extension to arbitrary open sets is standard and is omitted. Observe first that $T_n \cdot T_n = T_n$ and $T_n^{-1} = T_n$. Given a neighborhood $t_1 T_n t_2$ of $s_1 s_2$, then

$$s_1 \in t_1 T_n t_1^{-1} s_1,\ s_2 \in s_2 t_2^{-1} T_n t_2 \text{ and } t_1 T_n t_1^{-1} s_1 \cdot s_2 t_2^{-1} T_n t_2 \subseteq t_1 T_n t_2.$$

If $s^{-1} \in t_1 T_n t_2$, then

$$s \in t_2^{-1} T_n t_1^{-1} \text{ and } \left(t_2^{-1} T_n t_1^{-1}\right)^{-1} = t_1 T_n t_2.$$

Similarly, if $s_1 s_2 \in S(O_1, O_2)$, put $O = s_2(O_1)$. Then O is an open interval in some X_n, $s_1 \in S(O_1, O_2)$, $s_2 \in S(O_1, O)$,

$$S(O, O_2) \cdot S(O_1, O) \subseteq S(O_1, O_2).$$

If $s^{-1} \in S(O_1, O_2)$ then $s \in S(O_2, O_1)$ and $S(O_2, O_1)^{-1} = S(O_1, O_2)$. These remarks establish the continuity of multiplication and inversion, and show that T is a topological group.

To establish that this is a Hausdorff topology, let $t_1, t_2 \in T$, $t_1 \neq t_2$. Then there is an n and an $x \in X_n$ such that $t_1(x) \neq t_2(x)$. Pick disjoint open intervals O_1 and $O_2 \subseteq X_n$ containing $t_1(x), t_2(x)$, and an open interval $O \subseteq X_n$ containing x, such that $t_i(O) \subseteq O_i$ for $i = 1, 2$. The open sets $S(O, O_i)$ are disjoint and contain t_i, $i = 1, 2$.

We now show that each $(X,\mathcal{T}_i)$, $i=1,2$, is a flow under the action of T. To this end, let V_ε $(\varepsilon>0)$ be a $\mathcal{T}_1$-neighborhood of $t0=0$. Pick δ such that $tV_\delta \subseteq V_\varepsilon$ and then choose m such that $1/m<\varepsilon$. It is clear that $t\in tT_m$, $0\in U_m$ and $tT_mU_m\subseteq V_\varepsilon$. Since $V_\varepsilon \subseteq W(F,\varepsilon)$, the same argument works for $\mathcal{T}_2$ -neighborhoods of 0. Next assume $0\neq x\in X_n$ and $O\subseteq X_n$ is a neighborhood of tx. Pick an open interval $O_1\subseteq X_n$ containing x such that $t\left(O_1\right)\subseteq O$. Then $t\in S(O_1,O)$ and $S(O_1,O)O_1\subseteq O$. □

Corr 15 The key to showing that $(X,\mathcal{T}_1)$ is not compactifiable is the following result.

Proposition 30. *Let* $f:(X,\mathcal{T}_1)\to\mathbb{R}$ *be a* T*-uniformly continuous function with* $f(0)=0$. *Then for every* $\varepsilon>0$, $\{n:\sup|f(X_n)|\geq\varepsilon\}$ *is finite. In particular,* f *is* $\mathcal{T}_2$*-continuous.*

Proof. Suppose by way of contradiction that $N_1\subseteq\mathbb{N}$ is infinite, $\varepsilon>0$ and $\sup\left|f\left(X_n\right)\right|\geq\varepsilon$ for each $n\in N_1$. Also, by Prop. 7, $|f|$ is T-uniformly continuous, so we may assume that $f\geq 0$.

Since f is $\mathcal{T}_1$-continuous, there is an integer m such that $\sup f\left(U_m\right)<\varepsilon/2$. Put $\delta=\varepsilon/4m$. Since f is T-uniformly continuous, there exists a basic open neighborhood V of 1 in T such that

$$\left|f\left(tx\right)-f\left(x\right)\right|<\delta$$

for all $x\in X$, $t\in V$. By passing to a smaller neighborhood if necessary, we may assume that

$$V\subseteq\bigcap_{i=1}^{m}T_i.$$

The neighborhood V is determined by a finite number of neighborhoods of the form sT_ks^{-1} and a finite number of neighborhoods of the form $S(O_1,O_2)$. Since each of the homeomorphisms s involved in these neighborhoods is the identity on all but finitely many X_n's, and each $S(O_1,O_2)$ restricts the actions on only one X_n, we may discard a finite number of n's and obtain an infinite subset $N_2\subseteq N_1$ such that each of these s's is the identity on X_n and each $S(O_1,O_2)|_{X_n}=T\ |_{X_n}$ for all $n\in N_2$. So, if $n\in N_2$, $1\leq j\leq m$, and $t\in V$, Corr 16

$$t\left(U_j\cap X_n\right)=U_j\cap X_n.$$

For the remainder of the proof, fix $n\in N_2$. Let us regard the elements $x,y\in X_n$. Since $\sup f(U_m)<\varepsilon/2$, $f(1/m)\leq\varepsilon/2$. By continuity of f, there is an x, $1/m<x<1/(m-1)$, such that $f(y)\leq\varepsilon/2+\delta$ for all $y\leq x$. Next, let us consider y such that $x<y<1/(m-1)$. Since the behavior of $t \in V$ is "unrestricted" between $1/m$ and $1/(m-1)$, there is a $t \in V$ such that $tx = y$. In particular,

$$|f(y) - f(x)| = |f(tx) - f(x)| < \delta,$$

so $f(y) < f(1/m) + 2\delta$. By continuity, $f(1/(m-1)) \leq f(1/m) + 2\delta$. We have shown that

$$0 \leq y \leq \frac{1}{m-1} \Longrightarrow f(y) \leq \varepsilon/2 + 2\delta.$$

Repeating this argument for $2 \leq j \leq m-1$, it follows by finite induction that

$$0 \leq y \leq \frac{1}{m-j} \Longrightarrow f(y) \leq \varepsilon/2 + 2\delta j.$$

When $j = m-1$, we get that

$$0 \leq y \leq 1 \Longrightarrow f(y) \leq \varepsilon/2 + 2\delta(m-1) < \varepsilon,$$

so $\sup f(X_n) < \varepsilon$. This contradicts the assumption that $\sup f(X_n) \geq \varepsilon$. It is now routine to check that f is $\mathcal{T}_2$-continuous. □

We have now assembled the machinery necessary to complete our example.

Theorem 31. *$(X, \mathcal{T}_1)$ is not flow compactifiable. $(X, \mathcal{T}_2)$ is a compact flow which is the finest compactifiable quotient flow of $(X, \mathcal{T}_1)$. (The existence of such a finest compactifiable quotient flow was established in Corollary 25).*

Proof. It suffices to show (by property (3) of Theorem 24) that $C^T X$ does not determine the topology on X. Let $K = \{x \in X : ||x|| = 1\}$. K is a $\mathcal{T}_1$-closed set, not containing 0. If $C^T X$ determines the topology on X, there exist $f \in C^T X$ and $\varepsilon > 0$ such that $f(0) = 0$ and $f(x) \geq \varepsilon$ for all $x \in K$. But Proposition 30 shows that this is an impossibility. □

By a simple modification of this example we can exhibit a completely regular flow Y acted upon by the group T of Example 2 so that $C^T Y$ does not separate the points of Y. We sketch the details.

Let $Y = X \cup \{\infty\}$, where X is the space from Example 2. Topologize Y by declaring sets in $\mathcal{T}_1$ to be open while neighborhoods of ∞ are of the form $Y_n = \{\infty\} \cup \bigcup_{m=n}^{\infty} \{x \in X_m : x > 1/2\}$ for $n = 1, 2, \cdots$. Evidently, Y is a completely regular space. Extend the homeomorphisms in T to Y via $t\infty = \infty$. To verify that Y is a flow one needs only to check continuity of evaluation at $t\infty$. To do this take a neighborhood Y_n of $t\infty$. By passing to a larger n if necessary we may assume that t is the identity on $Y_n \cap X$. If $(t', y) \in T_2 t \times Y_n$, then $t'y \in Y_n$, as desired. It is now an immediate consequence of Proposition 30 that if $f \in C^T Y$, then $f(0) = f(\infty)$.

One might summarize the examples in this section as follows: There are flows X for which the interaction of T with open subsets of X is incompatible with the existence of some T-uniformly continuous functions. In Section 9 we examine the structures of open sets of X, and their interactions with T, which are compatible with T-uniformly continuous functions.

Examples 1 and 2 illustrate clearly the (apparent) additional structure when T is a topological group of actions. More precisely, in Example 1 we presented an example of a flow with no nonconstant T-uniformly continuous functions, while in Example 2, we were able to construct a noncompactifiable flow X with T a topological group. In this example there are enough T-uniformly continuous functions to separate the points of X. As we observed, it is easy to modify this example to produce two points which cannot be separated by any T-uniformly continuous function. But many questions are suggested by this disparity. We state only the most apparent of these here. Suppose that X is a flow with a topological group T of actions. Does there exist a nonconstant T-uniformly continuous function on X?

8. ONE POINT COMPACTIFICATIONS

In this section we investigate the question of when a flow X has a one-point flow compactification. A necessary condition is that X have a one-point space compactification, i.e., that X be locally compact. If X is locally compact we use ∞ to denote the additional point in its one-point compactification, and we use $\bar{X}$ to denote the compactification itself. The open neighborhoods of ∞ are the subsets of $\bar{X}$ whose complements are compact subsets of X.

First we must extend the actions on X to $\bar{X}$. The simplest way to do this is to have each action fix ∞, and that is the extension we consider here. But this requires that the actions be *perfect*. Recall that a continuous function $f : X \to Y$ is said to be *perfect* if it is closed and $f^{-1}y$ is compact for each $y \in Y$. A continuous function $f : X \to Y$ is always perfect when X is compact, and when Y is locally compact f is perfect if and only if $f^{-1}C$ is compact in X whenever C is compact in Y.

Lemma 32. *Let $t : X \to X$ be a continuous function on the locally compact space X. Then t can be continuously extended to $\bar{X}$ in such a way that the extension fixes ∞ if and only if t is perfect.* Corr 17

Proof. Let t be perfect, and let $\bar{t}$ be its extension to $\bar{X}$ by fixing ∞. To show $\bar{t}$ continuous at ∞, consider an open neighborhood U of ∞ in $\bar{X}$. Then $K = \bar{X} \setminus U$ is compact in X, hence $t^{-1}K = L$ is compact as well. Then $V = \bar{X} \setminus L$ is a neighborhood of ∞, and $tV \subseteq U$, i.e., $\bar{t}$ is continuous at ∞. Now suppose that $\bar{t} : \bar{X} \to \bar{X}$ is continuous. Then $\bar{t}$ is perfect, and its restriction t is also perfect because $tC = \bar{t}C$ for any closed subset $C \subseteq X$, and because $t^{-1}x = \bar{t}^{-1}x$ for all $x \in X$. □

Definition . *We say that* T acts perfectly on X *if* T *acts on* X *in such a way that each action is a perfect map.*

Example 1 shows that not every locally compact flow on which T acts perfectly is compactifiable. We seek to characterize those that are.

Proposition 33. *Suppose* X *is a locally compact flow on which* T *acts perfectly. Then* X *is compactifiable if and only if* $\bar{X}$ *is a flow compactification of* X*.*

Proof. Suppose Y is a flow compactification of X. The collapsing map
c, defined by $cx = x$ for all $x \in X$ and $cr = \infty$ for all points r in Corr 18
the remainder $R = Y \setminus X$, is continuous because X is an open subset of both $\bar{X}$ and Y. Let $e_{\bar{X}}$ and e_Y denote the evaluation maps, and let d be the map induced by ce_Y and the identity map on T, i.e., $d(t, y) = (t, cy)$. Then the following diagram commutes.

$$\begin{tikzcd}
T \times Y \arrow[r, "e_Y"] \arrow[d, "d"'] & Y \arrow[d, "c"] \\
T \times \bar{X} \arrow[r, "e_{\bar{X}}"'] & \bar{X}
\end{tikzcd}$$

Corr 19

To verify the continuity of $e_{\bar{X}}$ at a point of the form (t, ∞) consider an open neighborhood U of $t\infty = \infty$ in $\bar{X}$. Then $c^{-1}U$ is an open subset
of Y containing R, and so by the continuity of e_Y there are for each Corr 20
$r \in R$ open sets $U_r \in \mathcal{N}_r$ and $T_r \in \mathcal{N}_t$ such that $t_r r' \in U$ for all $t_r \in T_r$ and $r' \in U_r$. Since R is compact there is a finite subset $R_0 \subseteq R$ such that $\{U_r : r \in R_0\}$ covers R; set

$$T_0 = \bigcap_{R_0} T_r, \quad \text{and} \quad U_0 = \bigcup_{R_0} U_r.$$

Since $K = Y \setminus U_0$ is a compact subset of X, $V = \bar{X} \setminus K$ is a neighborhood of ∞ in $\bar{X}$. And for all $t' \in T'$ and $x \in V$ we have some $y \in U_0$ such that $cy = x$, with the result that

$$t'x = e_{\bar{X}}(t', x) = e_{\bar{X}} d(t', y) = ce_Y(t', y) \in U. ∎$$

Theorem 34. *Any locally compact flow acted upon by a topological group has a one-point flow compactification.*

Proof. Let X be a locally compact flow acted upon by a topological group T. Note first that the actions are homeomorphisms and hence perfect, so they extend continuously to $\bar{X}$ by fixing ∞. To show the evaluation map $e : T \times \bar{X} \to \bar{X}$ continuous at the point (t, ∞), consider an open neighborhood U of $t\infty = \infty$, say $U = \bar{X} \setminus K$ for compact $K \subseteq X$. Because $t^{-1}K$ is compact and X is locally compact, $t^{-1}K$ has a compact neighborhood L; let $W = \bar{X} \setminus L$ be the corresponding neighborhood of ∞. For each $x \in K$ find $T_x \in \mathcal{N}_{t^{-1}}$ and $V_x \in \mathcal{N}_x$ such that $t_x x' \in L$ for all $t_x \in T_x$ and $x' \in V_x$. Since K is compact, there is a finite subset $K_0 \subseteq K$ such that $\{V_x : x \in K_0\}$ covers K; set $T_0 = \bigcap_{K_0} T_x \in N_{t^{-1}}$ and $T_1 = T_0^{-1} \in \mathcal{N}_t$. Then for all $t_1 \in T_1$ and
Corr 21 all $x' \in K$ we have $t_1^{-1}x' \in L$, hence $t_1^{-1}K \subseteq L$. Therefore for all $t_1 \in T_1$ and $x \in W$ we have $t_1 x \in W$, since $t_1 x \in K$ would imply $x \in t_1^{-1}K \subseteq L$. ☐

The hypothesis of local compactness cannot be omitted from Theorem 34, as the examples of Section 7 show. We summarize the results of this section in the following theorem.

Theorem 35. *A noncompact flow X admits a one-point flow compactification in which each action fixes the point at infinity if and only if the following conditions obtain.*

1. *X is locally compact.*
2. *Each action is a perfect map.*
3. *X is compactifiable.*

And if T is a topological group then any locally compact flow admits such a one-point flow compactification.

We close this section with an observation.

Proposition 36. *Let X be a flow satisfying the conditions of Theorem 35, let Y be any compactification of X, and let $R = Y \setminus X$ be the remainder. Then R is a compact flow, and every compact flow arises in this manner.*

Proof. Each action on Y, being perfect on X, maps R into itself [19]. And R is compact because X is an open subset of Y by virtue of its local compactness. Thus R is a compact flow.

Now suppose that R is a given compact flow. Let $Y = R \times \bar{N}$ with actions defined by $t\,(r, n) = (tr, n)$ for $n \in \bar{N}$, and let $X = R \times N$. ☐

9. T-SCALES

A *scale* is a collection $S = \{U_q\}$ of open subsets of a space X, indexed by the rational interval $[0,1]_{\mathbb{Q}}$ in such a way that $\operatorname{cl} U_q \subseteq U_p$ for all $q < p$ [14, IV 1.4]. Scales arise naturally as

$$\left\{g^{-1}(-\infty, p) : p \in [0,1]_{\mathbb{Q}}\right\}$$

for functions $g \in CX$ such that $0 \le g \le 1$. In fact, from a scale S one can construct such a function by setting

$$gx = \bigwedge \{q : x \in U_q\},$$

with the understanding that $\bigwedge \emptyset = 1$, and g will not only be continuous, but will also have the feature that for every $r \in [0,1]$

$$g^{-1}(-\infty, r) = \bigcup \{U_q : q < r\}.$$

We will refer to g as *the function associated with the scale S.* Corr 22

Definition . *For subsets U and V of a flow X, let us say that U is* T-disjoint *from V provided that for all $t \in T$ there is a neighborhood $T_t \in \mathcal{N}_t$ such that for all $s \in T_t$ it is true that*

$$s^{-1}U \cap t^{-1}V = t^{-1}U \cap s^{-1}V = \emptyset.$$

We say that U is T-contained *in V, or that V* T-contains *U, provided that U is T-disjoint from $X \setminus V$. That is, U is T-contained in V if and only if for all $t \in T$ there is a neighborhood $T_t \in \mathcal{N}_t$ such that for all $s \in T_t$ we have*

$$s^{-1}U \subseteq t^{-1}V \quad \text{and} \quad t^{-1}U \subseteq s^{-1}V.$$

Note that if the topology on T is based at 1 then these conditions need only be verified at $t = 1$, and if T is a topological group then each pair of conditions reduces to a single condition.

Definition . *A* T-scale *on a flow X is a scale $S = \{U_q\}$ such that U_q is T-contained in U_p for all $q < p$.*

Theorem 37. *Any T-uniformly continuous function g on a flow X gives rise to a T-scale, and conversely, the function associated with any T-scale is T-uniformly continuous.*

Proof. Suppose $g \in C^T X$, set $U = g^{-1}(-\infty, q)$ and $V = g^{-1}(-\infty, p)$ for some $q < p$ in $[0,1]_{\mathbb{Q}}$, consider $t \in T$, and set $\epsilon = p - q$. Then use condition (2) of Theorem 6 to find $T_t \in \mathcal{N}_t$ such that $|gt - gs| < \epsilon$ for all $s \in T_t$. Thus

$$\begin{aligned} x &\in t^{-1}U = t^{-1}g^{-1}(-\infty, q) \iff gtx < q \implies gsx < p \iff \\ x &\in s^{-1}g^{-1}(-\infty, p) = s^{-1}V, \end{aligned}$$

and similarly $s^{-1}U \subseteq t^{-1}V$. This shows that $\left\{g^{-1}(-\infty, q) : q \in [0,1]_{\mathbb{Q}}\right\}$ is a T-scale.

Conversely suppose we are given a T-scale $S = \{U_q\}$. To verify that the associated function g is T-uniformly continuous, consider $\epsilon \in (0,1)_{\mathbb{Q}}$ and $t \in T$. We claim there is some $T_t \in \mathcal{N}_t$ for which

$$s^{-1}U_q \subseteq t^{-1}U_{q+\epsilon} \quad \text{and} \quad t^{-1}U_q \subseteq s^{-1}U_{q+\epsilon}$$

for all $s \in T_t$ and all $q \in [0,1]_{\mathbb{Q}}$ for which $q + \epsilon \in [0,1]_{\mathbb{Q}}$. For if n is a positive integer greater than $2/\epsilon$ then for each k, $0 \leq k \leq n-1$, there is some $T_k \in \mathcal{N}_t$ such that

$$s^{-1}U_{\frac{k}{n}} \subseteq t^{-1}U_{\frac{k+1}{n}} \quad \text{and} \quad t^{-1}U_{\frac{k}{n}} \subseteq s^{-1}U_{\frac{k+1}{n}}.$$

Then for all $s \in T_t = \bigcap_0^{n-1} T_k \in \mathcal{N}_t$ and all $q \in [0,1]_{\mathbb{Q}}$ such that $q + \epsilon \in [0,1]_{\mathbb{Q}}$ we have

$$s^{-1}U_q \subseteq s^{-1}U_{\frac{k}{n}} \subseteq t^{-1}U_{\frac{k+1}{n}} \subseteq U_{q+\epsilon},$$

and likewise $t^{-1}U_q \subseteq s^{-1}U_{q+\epsilon}$, where k is the least integer such that $k/n \geq q$. This proves the claim, which in turn gives for $s \in T_t$ Corr 23

$$\begin{aligned} gtx &= \bigwedge \{q : tx \in U_q\} = \bigwedge \left\{q : x \in t^{-1}U_q\right\} \\ &\geq \bigwedge \left\{q : x \in s^{-1}U_{q+\epsilon}\right\} = \bigwedge \left\{q + \epsilon : sx \in U_{q+\epsilon}\right\} - \epsilon \\ &= gsx - \epsilon. \end{aligned}$$

A similar argument yields $gsx \geq gtx - \epsilon$. It follows that g is T-uniformly continuous. □

Definition . *Two subsets A and B of a flow X are said to be T-*completely separated *if there is a T-uniformly continuous function g on X which is* 0 *on (all points of) A and* 1 *on B.*

We remark that a flow is compactifiable if and only if each of its points is T-completely separated from every closed set not containing it. Evidently the foregoing considerations establish the following facts.

Proposition 38. *Two subsets A and B of a flow X are T-completely separated if and only if there is a T-scale $S = \{U_q\}$ such that $A \subseteq U_0$ and $B \cap U_1 = \emptyset$.*

Definition . *Suppose that X is a flow. A* T-zero set (T-cozero set) *of X is one of the form* $\operatorname{zero} g$ $(\operatorname{coz} g)$ *for $g \in C^T X$.*

Proposition 39. *A subset U of a flow X is a T-cozero set if and only if it is of the form $\bigcup_{q<1} U_q$ for some T-scale $S = \left\{U_q : q \in [0,1]_{\mathbb{Q}}\right\}$.*

Proposition 40. *The collection of T-zero sets of a flow X is closed under finite union, countable intersection, and inverse image under continuous flow maps.*

Proof. Suppose $\{Z_n\}$ is a sequence of T-zero sets of the flow X, say $Z_n = \operatorname{zero} g_n$ for $0 \leq g_n \in C^T X$. Then the closure of $C^T X$ under infima implies that $Z_i \cup Z_j = \operatorname{zero}(g_i \wedge g_j)$ is a T-cozero set of X. Likewise the fact that $h_n = \sum_{i=1}^n g_i/2^i$ converges uniformly to $h = \sum_{i=1}^\infty g_i/2^i$, and that $C^T X$ is uniformly closed, implies that $\bigcap Z_n$ is a T-zero set of X, namely $\operatorname{zero} h$. And the closure of T-zero sets under inverse flow maps is a consequence of Proposition 8. $\square$

Here is an example which shows that, in contrast to the classical Corr 24
(no-action) situation [17, 1.9 (f)], T-disjoint T-zero sets need not be T-completely separated. We sketch the details, which are similar to those of Example 2.

Example 3. *A flow X with two T-disjoint T-zero sets which are not T-completely separated.*

Let X be the subspace of $\mathbb{R}^2$ defined by

$$X = \left\{ (n, r) : n \in \mathbb{N},\ r = 0 \text{ or } \pm\frac{1}{m} \text{ for some } m \in \mathbb{N} \right\}.$$

Let us agree that mention of r automatically restricts r to one of the above values.

An action $t \in T$ satisfies

- t is a homeomorphism of X.
- t moves only finitely many points. (It follows that $t(n, 0) = (n, 0)$ for every $n \in \mathbb{N}$.)
- t takes points in the upper half plane to the upper half plane and points in the lower half plane to the lower half plane.

It is clear that T is a group which acts on X. In order to make X into a flow, we provide T with a topology as follows. As in Example 2 we describe two types of neighborhoods of $1 \in T$.

- For a finite $X_0 \subseteq X$, let $T(X_0) = \{t \in T : tx = x \text{ for all } x \in X_0\}$.
- Next, for $m, n \in \mathbb{N}$, let
  $T(n,m) = \left\{t : t(n,r) = (n,r) \text{ for } r = 0 \text{ or } r = \frac{1}{k} \text{ for } |k| \geq m\right\}$.

Note that $T(X_0)$ is a subgroup of T and that $T(X_0)^t = tT(X_0)t^{-1} = T(tX_0)$. Also, each $T(n,m)$ is a subgroup of T. It is now easy to check that these sets form a subbase of neighborhoods of 1 for a Hausdorff group topology on T and that X is a flow. The following statements are also fairly straightforward.

(1) $\beta^T X = \bar{X}$, the one-point flow compactification of X. That is, X has only one flow compactification. To see this, we claim that if f is a T-uniformly continuous function on X and $\varepsilon > 0$ is given, then there is an N such that if $m, n \geq N$, then $|f(n,r) - f(m,s)| < \varepsilon$. Indeed, given such an ε, let T_1 be a neighborhood of 1 such that $|ftx - fx| < \varepsilon$ for $t \in T_1$, $x \in X$. It follows that there is an N such that if $m, n \geq N$ and r, s are both positive or both negative, there is a $t \in T_1$ such that $t(n,r) = (m,s)$. This fact, the T-uniform continuity and the continuity of f now immediately yield the claim. Thus, if f is T-uniformly continuous and r_n is arbitrary, $\lim_{n\to\infty} f(n, r_n)$ exists. In other words, the only flow compactification of X is its one-point flow compactification.

(2) The sets $\{(n,1) : n \in \mathbb{N}\}$ and $\{(n,-1) : n \in \mathbb{N}\}$ are T-disjoint T-zero sets in X. (Since the actions in T respect both half planes, the neighborhood T of any t gives the T-disjointness condition. If $\varepsilon = +1$ or -1, the function $f : X \to \mathbb{R}$ given by $f(n,r) = 0$ if $r = \varepsilon$ and $1/n$ otherwise is T-uniformly continuous and has $\{(n,\varepsilon) : n \in \mathbb{N}\}$ as its zero set.)

(3) These sets are *not* T-completely separated. To see this, let $g : X \to \mathbb{R}$ be a T-uniformly continuous function such that $g(n,-1) = 0$ for all n. Then, by (1), its extension $\bar{g} : \bar{X} \to \mathbb{R}$ must satisfy $\bar{g}(\infty) = 0$, so $g(n,1)$ cannot be 1 for all $n \in \mathbb{N}$.

We conclude from this example that no disjointness condition on T-zero sets, short of the condition of Proposition 38, is likely to be sufficient to ensure their T-complete separation.

Definition . *A subflow $A \subseteq X$ is* C^T-embedded *if for every $g \in C^T A$ there is some $h \in C^T X$ such that $h|A = g$.*

Proposition 41. *The only compactification in which a compactifiable flow X is C^T-embedded is $\beta^T X$.*

Proof. Each $g \in C^T X$ extends to $\bar{g} \in C^T Y^T X$ by Proposition 14, which is to say that X is C^T-embedded in $\beta^T X$. Now consider a compact flow Y in which X is C^T-embedded. We may assume that X is dense in Y, by exchanging Y for the closure of X if necessary. Then from the embedding i of X in Y we get by Theorem 23 a flow map $f = \beta^T i : \beta^T X \to Y$ such that $i = f\beta^T_X$. In fact, f is the flow map produced by Theorem 21 from the $T\ell$-map k which takes each $g \in G = C^T X$ to its unique extension $kg \in C^T Y$. Then f is injective because kG separates the points of Y, and f is surjective because k is injective. □

Urysohn's Extension Theorem holds for flows. We refer the interested reader to [17, 1.9 (h), (i)].

Theorem 42. *A subflow A is C^T-embedded in a flow X if and only if any T-completely separated subsets of A are T-completely separated in X.*

Proof. The proof for the classical (no action) theorem, which depends only on the uniform closure of CX, generalizes without problem to $C^T X$ because the latter is also uniformly closed. $\square$

Corollary 43. *In a compactifiable flow any compact subset is T-completely separated from any closed set disjoint from it, and any compact subflow is C^T-embedded.*

10. T-NORMAL FLOWS

Definition . *A flow is* T-normal *if every pair of T-disjoint closed sets is T-contained in a pair of open sets whose closures are T-disjoint. (It is enough that for T-disjoint closed sets C_0 and C_1 there is an open set U_0 which T-contains C_0 and whose closure is T-disjoint from C_1.)*

Compact Hausdorff flows are T-normal, at least if the topology on T is based at 1.

Definition . *A flow X is* T-Hausdorff *if for every pair of distinct points x_i in X there are open sets U_i containing x_i such that the closures of the U_i's are T-disjoint.*

Lemma 44. *Suppose the topology on T is based at 1. Then a regular flow is T-Hausdorff.*

Proof. Given distinct $x_i \in X$ find disjoint open sets V_i containing x_i. Then use the continuity of evaluation to find open sets $W_i \subseteq V_i$ and neighborhood T_1 of 1 such that $t_1 x'_i \in V_i$ for all $x'_i \in W_i$ and $t_1 \in T_1$. Finally use the regularity of X to find open sets U_i containing x_i such that $\operatorname{cl} U_i \subseteq W_i$. We claim that $\operatorname{cl} U_1$ is T-disjoint from $\operatorname{cl} U_2$. For if $x \in \operatorname{cl} U_1 \subseteq W_1$ then for any $t_1 \in T_1$ we would have $t_1 x \in V_1$, hence $x \notin t_1^{-1} V_2 \supseteq t_1^{-1} \operatorname{cl} U_2$. Likewise $t_1^{-1} \operatorname{cl} U_1 \cap \operatorname{cl} U_2 = \emptyset$. $\square$

The next result is easily proven by a modification of the argument for the classical (no-action) case. The only wrinkle is that one must intersect a finite number of neighborhoods of a given action.

Proposition 45. *A compact T-Hausdorff flow is T-normal. In particular, if the topology of T is based at 1 then every compact Hausdorff flow is T-normal.*

T-normality is exactly the condition required to construct T-scales inductively.

Theorem 46. *A flow X is T-normal if and only if every pair of T-disjoint closed subsets of X is T-completely separated.*

Proof. Suppose that C_0 and C_1 are T-completely separated by $g \in C^T X$, i.e., g is 0 on C_0 and 1 on C_1. Set $U_0 = g^{-1}(-\infty, 1/2)$. To show that U_0 T-contains C_0 consider $t \in T$, and let $T_t \in \mathcal{N}_t$ be such that $|gs - gt| < 1/2$ for all $s \in T_t$. Then

$$\begin{aligned} x \in s^{-1}C_0 &\implies sx \in C_0 \implies gsx = 0 \implies gtx < 1/2 \implies \\ tx \in U_0 &\implies x \in t^{-1}U_0, \end{aligned}$$

and $t^{-1}C_0 \subseteq s^{-1}U_0$ similarly. To show that $\operatorname{cl} U_0$ is T-disjoint from C_1 consider $t \in T$, and for this t choose T_t as before. Then any point x in $t^{-1} \operatorname{cl} U_0$ satisfies $gtx \le 1/2$, while any x in $s^{-1}C_1$ satisfies $sx = 1$. Thus a point which is simultaneously in both sets would violate the condition $|gt - gs| < 1/2$, implying that the sets are disjoint. Likewise $s^{-1} \operatorname{cl} U_0 \cap t^{-1}C_1 = \emptyset$. That is, the sets are T-disjoint.

Suppose C_0 and C_1 are T-disjoint subsets of the T-normal flow X. Let U_0 be an open set T-containing C_0 whose closure is T-disjoint from C_1, and let $U_1 = X \setminus C_1$. Then because $\operatorname{cl} U_0$ is T-contained in U_1, the T-normality of X allows the inductive construction of a T-scale by the usual back-and-forth argument [17, 1.10]. And the associated function $g \in C^T X$ is 0 on C_0 and 1 on C_1. □

Corollary 47. *A T-normal flow is compactifiable, and every closed subflow is C^T-embedded.*

Proof. The fact that a T-normal flow X is compactifiable follows from Theorem 24. Consider now a closed subflow $C \subseteq X$. If A_1 and A_2 are two T-completely separated subsets of C then so are $\operatorname{cl} A_1$ and $\operatorname{cl} A_2$, from which it follows that the latter are T-disjoint in C and hence also in X. It follows from Theorem 46 that A_1 and A_2 are T-completely separated in X. The result then follows from Theorem 42. □

Example 3 is compactifiable but not T-normal. Moreover, its only proper closed subflows are the sets $\{(n, r) : n \in \mathbb{N},\ r \ge 0 \text{ (or } r \le 0)\}$ and subsets of $\{(n, 0) : n \in \mathbb{N}\}$. Clearly, each of these is C^T-embedded. Thus neither consequence of Corollary 47 implies T-normality. This differs from the classical situation inasmuch as any space in which every closed subspace is C^*-embedded is normal [17, 1.10 (g)].

11. Compactifications of X versus topologies on T

The topology on T determines which of the continuous functions on a flow X are T-uniformly continuous, and thereby determines the largest flow compactification $\beta^T X$ of X. We close this article by pointing out that this dependence is reversible to some extent, i.e., that a flow compactification determines the T-uniformly continuous functions on X, which in turn implies a coarse bound on the topology of T. Thus there is a kind of duality between the compactifications of a flow X and the topologies on T.

Proposition 48. *Let Y be a compact space on which T acts and let $G = CY$. Then the coarsest monoid topology on T with respect to which Y is a flow has neighborhoods of $t \in T$ of the form*

$$T(t, g, \epsilon) = \{t' : |gt' - gt| < \epsilon\},$$

for $g \in G$ and $\epsilon > 0$.

Proof. These neighborhoods must be present in any topology which makes the functions of CY T-uniformly continuous. We must show that they render the monoid multiplication continuous. For $t_i \in T$ and neighborhood $T(t_1 t_2, g, \epsilon)$, consider $t_1' \in T(t_1, g, \epsilon/2)$ and $t_2' \in T(t_2, gt_1, \epsilon/2)$. Then Corr 25

$$|gt_1't_2' - gt_1t_2| \le |gt_1't_2' - gt_1t_2'| + |gt_1t_2' - gt_1t_2| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \blacksquare$$

Consider, for instance, Example 1. Each action on $\mathbb{N}$ extends to the one point compactification $\bar{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$ by simply fixing ∞, and the coarsest topology on T which makes $\bar{\mathbb{N}}$ a flow has neighborhoods of $t \in T$ of the form

$$T(t, m) = \{t' : \forall n \le m \quad t'(n) = t(n), \text{ and } \forall\ n \ge m \quad t'(n) \ge m\}.$$

This topology is strictly finer than the one given in Example 1, and that fact "explains" why $\mathbb{N}$ is not compactifiable with respect to the coarser topology on T. When T is equipped with the finer topology, however, it is not only compactifiable, but $\bar{\mathbb{N}}$ is $\beta^T\mathbb{N}$, the largest compactification of $\mathbb{N}$. Thus $\bar{\mathbb{N}}$ corresponds to the finer topology. Corr 26 in bibliography

References

[1] M. Anderson and T. Feil. *Lattice-Ordered Groups, an Introduction. Reidel Texts in the Mathematical Sciences*, Reidel, Dordrecht, 1988.

[2] J. Auslander. *Minimal Flows and their Extensions.* North-Holland, 1988.

[3] R. N. Ball and A. W. Hager. On the localic Yosida representation of an archimedean lattice ordered group with weak unit. *J. Pure Appl. Alg.*, 70:17–43, 1991.

[4] R. N. Ball and J. N. Hagler. The Gleason cover of a flow. *Papers on General Topology and Applications: Tenth Summer Conference at Amsterdam*, K.P. Hart and E. Coplakova, Eds., New York Academy of Sciences. to appear.

[5] R. N. Ball and J. N. Hagler. T-Boolean algebras. in preparation.

[6] R. N. Ball and J. N. Hagler. Localic Flows. in preparation.

[7] R. N. Ball and J. N. Hagler. Actions on Archimedean lattice-ordered groups with strong unit. Proceedings of the Conference on Ordered Algebra, Curaçao, Netherlands Antilles. submitted.
[8] A. Bigard, K. Keimel, and S. Wolfenstein. *Groupes et Anneaux Reticulés. Lecture Notes in Pure and Applied Mathematics 608*, Springer Verlag, 1977.
[9] R. Ellis. *Lectures on Topological Dynamics.* Benjamin, 1969.
[10] H. Furstenberg. *Recurrence in Ergodic Theory and Combinatorial Number Theory.* Princeton University Press, 1981.
[11] L. Gillman and M. Jerison. *Rings of Continuous Functions.* Van Nostrand, Princeton, 1960. reprinted as Springer-Verlag Graduate Texts 43, Berlin-Heidelberg-New York, 1976.
[12] A. M. W. Glass and W. C. Holland, editors. *Lattice-Ordered Groups, Advances and Techniques*, Kluwer, Dordrecht, Boston, London, 1989.
[13] H. Herrlich and G. E. Strecker. *Category Theory.* Allyn and Bacon, Inc., 1973.
[14] P. T. Johnstone. *Stone Spaces.* Cambridge University Press, Cambridge, 1982.
[15] J. L. Kelley. *General Topology. Graduate Texts in Mathematics*, Springer-Verlag, 1955.
[16] W. Luxemburg and A. Zaanen. *Riesz Spaces.* North-Holland, Amsterdam, 1971.
[17] J. R. Porter and R. G. Woods. *Extensions and Absolutes of Hausdorff Spaces.* Springer-Verlag, 1988.
[18] J. C. S. P. van der Woude. *Topological Dynamix.* CWI Tracts, 1986.
[19] R. C. Walker. *The Stone-Čech Compactification.* Springer-Verlag, 1974.
[20] K. Yosida. On the representation of the vector lattice. *Proc. Imp. Acad. Tokyo*, 18:339–342, 1942.

(Ball and Hagler) DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE, UNIVERSITY OF DENVER, DENVER, COLORADO 80208, U.S.A.

E-mail address, Ball: `rball@@du.edu`
E-mail address, Hagler: `jhagler@@du.edu`