SIMILARITY OF NESTS IN L_p, $1 \leq p \neq 2 < \infty$

by

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ABSTRACT. In this note we prove that some aspect of the similarity theory for the Volterra nest in $L_p(0,1)$ for $1 < p \neq 2 < \infty$ is like that for $p = 1$; we thus answer a question from [ALWW].

1. INTRODUCTION.

A nest $\mathcal{N}$ in a Banach space X is a totally ordered family of closed subspaces closed under intersection and closed unions and containing 0 and X. The nest algebra induced by $\mathcal{N}$ is the set of all $T \in B(X)$ that leave invariant every element of $\mathcal{N}$; i.e., $TN \subset N$ for every $N \in \mathcal{N}$.

The main example for our purpose is the Volterra nest in $L_p(0,1)$, $1 \leq p < \infty$; where $\mathcal{N} = \{N_t : 0 \leq t \leq 1\}$ and N_t is the set of those functions $f \in L_p(0,1)$ that have support contained in $[0,t]$. For $p = 2$ the similarity theory tells us that this is the canonical example of a continuous nest (see [L]).

Another consequence of the similarity theory (see [D] and [L]) says the following: If $\phi : [0,1] \to [0,1]$ is strictly increasing and onto, then there exists $T \in B(L_2(0,1))$ invertible such that for every $t \in [0,1]$, $TN_t = N_{\phi(t)}$.

The last question was considered by Allen, Larson, Ward and Woodward ([ALWW]) for $p = 1$ where they proved that such a T exists if and only if both ϕ and ϕ^{-1} are absolutely continuous. They also asked the same question for $1 < p \neq 2 < \infty$, and gave some partial answer.

2. THE MAIN RESULT.

In this section we are going to show that the similarity theory for the Volterra nest acting on $L_p(0,1)$, $1 \leq p \neq 2 \leq \infty$ is like that of $L_1(0,1)$ (see [ALWW]).

THEOREM 1. Let $1 < p \neq 2 < \infty$ and $\phi : [0,1] \to [0,1]$ strictly increasing and onto. There exists $T : L_p(0,1) \to L_p(0,1)$ invertible satisfying $TN_t = N_{\phi(t)}$ for every $0 \leq t \leq 1$

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if and only if ϕ and ϕ^{-1} are absolutely continuous.

As building tools we are going to use the Haar basis $\{h_{n,i}\}_{i=1,}^{2^n}{}_{n=0}^{\infty}$ and the Rademacher functions $\{r_n\}_{n=0}^{\infty}$ where

$$h_{n,i} = \chi_{[\frac{i-1}{2^n},\frac{2i-1}{2^{n+1}}]} - \chi_{[\frac{2i-1}{2^{n+1}},\frac{i}{2^n}]} \qquad \text{and}$$
$$r_n = \sum_{i=1}^{2^n} h_{n,i}$$

The main property we use is the unconditionality of the Haar functions in $L_p(0,1)$ for $1 < p < \infty$. This allows us to use the "square function" instead of the "absolute value" as in [ALWW]. This technique was developed in [JMST] for a similar reason. We also use the fact that $\overline{\text{span}}\{r_n\} \subset L_p(0,1)$ is isomorphic to ℓ_2 for $0 < p < \infty$.

The σ–algebra generated by de diadic intervals $\Delta_{n,i} = [\frac{i-1}{2^n}, \frac{i}{2^n}]$, $1 \le i \le 2^n$, is denoted by $\mathcal{A}_n$. Clearly $\operatorname{supp} h_{n,i} \subset \Delta_{n,i}$.

PROOF OF THE THEOREM. It is enough to prove that ϕ^{-1} is absolutely continuous. That ϕ is absolutely continuous will follow by considering T^{-1}.

It is enough to prove Theorem 1 for $1 < p < 2$. The other side follows from duality.

One has to show only the "only if" part of the theorem. For if ϕ and ϕ^{-1} are absolutely continuous then $Tf(x) = f(\phi^{-1}(x))((\phi^{-1})'(x))^{\frac{1}{p}}$ has the desired property.

Assume that $T : L_p(0,1) \to L_p(0,1)$ is invertible and satisfies $TN_t = N_{\phi(t)}$ for every $0 \le t \le 1$.

The first step is to replace T by a "better" operator with the same properties.

Notice that $T : (\sum_{i=1}^{2^n} \oplus L_p(\Delta_{n,i}))_p \to (\sum_{i=1}^{2^n} \oplus L_p(\phi(\Delta_{n,i})))_p$ is "upper triangular" with respect to this decomposition. Take T_n to be the "diagonal"; i.e.,

$$T_n = \sum_{i=1}^{2^n} P_{\phi(\Delta_{n,i})} T P_{\Delta_{n,i}},$$

where for $A \subset [0,1]$, P_A is the projection in $L_p(0,1)$ that sends f to $\chi_A f$.

This operator was used in [ALWW]. It was shown there that for $p = 1$ it satifies $T_n N_t = N_{\phi(t)}$ for every t, $\|T_n\| \le \|T\|$ and $\|T_n^{-1}\| \le \|T^{-1}\|$; but the results extend easily for $0 < p < \infty$. Moreover, if a function f is supported on $\Delta_{n,i}$ then $T_n f$ is supported on $\phi(\Delta_{n,i})$. Equivalently, For every i, n, $1 \leq i \leq 2^n$,

$$P_{\phi(\Delta_{n,i})} T_n = T_n P_{\Delta_{n,i}}. \tag{1}$$

Let $1 < p < 2$. For every n define

$$\tilde{v}_n = \left(\sum_{i=1}^{2^n} |T h_{n,i}|^2 \right)^{\frac{1}{2}}.$$

CLAIM: $\{\tilde{v}_n^p\}_n$ is equi-integrable.

The claim was proved in [JMST] page 265. For completeness we will give a proof of it later.

We are going to use a set of functions smaller than the $\{\tilde{v}_n\}_n$'s that copy the behaviour of ϕ. Define

$$v_n = \left(\sum_{i=1}^{2^n} |T_n h_{n,i}|^2 \right)^{\frac{1}{2}}.$$

If we set $C = [0,1] \setminus \phi(\Delta_{n,i})$, then

$$\begin{aligned} T h_{n,i} &= P_{\phi(\Delta_{n,i})} T h_{n,i} + P_C T h_{n,i} \\ &= T_n h_{n,i} + P_C T h_{n,i} \end{aligned}$$

Since the latter two are disjoint we have that $|T_n h_{n,i}| \leq |T h_{n,i}|$. This implies that $\{v_n^p\}_n$ is equi-integrable. Notice also that for $i \neq j$, $T_n h_{n,i}$ and $T_n h_{n,j}$ are disjoint functions; hence,

$$v_n = \left(\sum_{i=1}^{2^n} |T_n h_{n,i}|^2 \right)^{\frac{1}{2}} = \sum_{i=1}^{2^n} |T_n h_{n,i}| = \left| T_n \left(\sum_{i=1}^{2^n} h_{n,i} \right) \right| = |T_n r_n|.$$

Let $g \in L_1(0,1)$ be a weak limit of $\{|T_n r_n|^p\}$. If $A \in \mathcal{A}_n$, the σ-algebra generated by $\{\Delta_{n,i}\}_{i=1}^{2^n}$, then using (1) we have

$$\int_{\phi(A)} |T_n r_n|^p \, dm = \int_0^1 |P_{\phi(A)} T_n r_n|^p \, dm = \int_0^1 |T_n (P_A r_n)|^p \, dm.$$

Since $\int_0^1 |P_A r_n|^p \, dm = m(A)$, then

$$\frac{1}{\|T_n^{-1}\|^p} m(A) \leq \int_{\phi(A)} |T_n r_n|^p \, dm \leq \|T_n\|^p m(A).$$

Therefore, if $A \in \bigcup_n \mathcal{A}_n$, we have that

$$\frac{1}{\|T^{-1}\|^p} m(A) \leq \int_{\phi(A)} g \, dm \leq \|T\|^p m(A). \tag{2}$$

Which clearly implies that ϕ^{-1} is absolutely continuous. ∎

REMARK. (1) It follows from (2) that

$$\frac{1}{\|T^{-1}\|^p}(\phi^{-1})'(x) \leq g(x) \leq \|T\|^p(\phi^{-1})'(x) \quad \text{m–a.e.}$$

Just take $A = (\phi^{-1}(x), \phi^{-1}(x+h))$, divide (2) by h and take the limit. This tells us that the operator $T_g : L_1(0,1) \to L_1(0,1)$ induced by g looks like the "natural" one. Moreover, it is not hard to see it is also invertible.

(2) The only property used was the existence of a family of functions $\{s_n\}_n$, $|s_n| \equiv 1$ for which $\{|T_n s_n|^p\}_n$ is equi-integrable. For $p = 1$ this is very easy to obtain, since for every $s \in L_1$, $|T_n s|$ is dominated by $|T||s|$, where $|T|$ is the absolute value of T. In particular, if one takes $s = \chi_{(0,1)}$, then $\{|T_n s|\}_n$ is equi-integrable. This recovers Theorem 4.4 of [ALWW].

(3) We also obtain new information for $p = 2$. If $\{|\tilde{v}_n|^2\}_n$ is equi-integrable, then T acts absolutely continuous. This last condition might be very hard to check, but it has the advantage that it does not make any reference to the particular nests.

PROOF OF THE CLAIM. ([JMST] page 265). The main tool we use is that if $\{f_i\} \subset L_p(0,1)$ is an unconditional basic sequence, then there exists a constant $C_p > 0$ such that for any $(a_i)_i$,

$$\frac{1}{C_p}\left\|\sum a_i f_i\right\|_p \leq \left\|\left(\sum |a_i|^2 |f_i|^2\right)^{\frac{1}{2}}\right\|_p \leq C_p \left\|\sum a_i f_i\right\|_p.$$

Since $\{h_{n,i}\}_{i=1,n=0}^{2^n,\infty}$ is an unconditional basis in $L_p(0,1)$ for $1 < p < 2$, and T is an isomorphism, then $\{Th_{n,i}\}$ is an unconditional basis. Assume that $\{\tilde{v}_n^p\}$ is not equi-integrable. Then we can find $\epsilon > 0$ and disjoint sets $\{A_k\}$ which satisfy

$$\left(\int_{A_k} \tilde{v}_{n_k}^p \, dm\right)^{\frac{1}{p}} \geq \epsilon.$$

Thus for every $(a_k)_k \in \ell_2$,

$$\Big(\sum_k |a_k|^2\Big)^{\frac{1}{2}} = \Big\|\sum_k a_k r_{n_k}\Big\|_2 \geq \Big\|\sum_k a_k r_{n_k}\Big\|_p$$
$$= \Big\|\sum_{k=1}^{\infty} a_k \sum_{i=1}^{m_k} h_{n_k,i}\Big\|_p \geq \|T\|^{-1}\Big\|\sum_{k=1}^{\infty} a_k \sum_{i=1}^{m_k} Th_{n_k,i}\Big\|_p$$
$$\geq \|T\|^{-1} C_p^{-1}\Big(\int_0^1 (\sum_k |a_k|^2 \tilde{v}_{n_k}^2)^{\frac{p}{2}} dm\Big)^{\frac{1}{p}}$$
$$\geq \|T\|^{-1} C_p^{-1}\Big(\sum_k \int_{A_k} |a_k|^p \tilde{v}_{n_k}^p dm\Big)^{\frac{1}{p}}$$
$$\geq \|T\|^{-1} C_p^{-1}\epsilon\Big(\sum_k |a_k|^p\Big)^{\frac{1}{p}}.$$

Since we assumed that $1 < p < 2$ this is contradiction. ∎

REMARKS. (1) The last step was the only place where we used that $p \neq 2$.

(2) The previous proof works also in more general situations. If ν and μ are two measures with no atoms and support $[0,1]$, $N_t \subset L_p(\nu)$ is the set of functions supported on $[0,t]$ and $M_t \subset L_p(\mu)$ is the set of functions supported on $[0,t]$. Then we can find $T : L_p(\nu) \to L_p(\mu)$ invertible, satisfying $TN_t = M_t$ for every $0 \leq t \leq 1$ if and only if ν and μ are mutually absolutely continuous with respect to each other. To see this, just take the Haar sistem of $L_p(\nu)$ and repeat the proof.

(3) G. Schechtman pointed out to us that the proof of the claim holds also for $0 < p \leq 1$; thus, extending Theorem 1 to those values. The reason for this is that for any sequence of signs $\epsilon_{n_k,i}$ one has that $\{\sum_{k=0}^{m_k} h_{n_k,i}\}_k \equiv \{\sum_{i=1}^{m_k} \epsilon_{n_k,i} h_{n_k,i}\}_k$, where the equivalence is in distributions; hence,

$$\Big\|\sum_{k=0}^{\infty} a_k \sum_{i=1}^{m_k} h_{n_k,i}\Big\|_p = \Big\|\sum_{k=0}^{\infty} a_k \sum_{i=1}^{m_k} \epsilon_{n_k,i} h_{n_k,i}\Big\|_p$$
$$\geq \|T\|^{-1}\Big\|\sum_{k=0}^{\infty} a_k \sum_{i=1}^{m_k} \epsilon_{n_k,i} Th_{n_k,i}\Big\|_p.$$

And after taking the average with respect to $\{\epsilon_{n_k,i}\}$, we apply Khintchine's inequality to get

$$\Big\|\sum_{k=0}^{\infty} a_k \sum_{i=1}^{m_k} h_{n_k,i}\Big\|_p \geq \|T\|^{-1} A_p\Big(\int_0^1 (\sum_k |a_k|^2 \tilde{v}_{n_k}^2)^{\frac{p}{2}} dm\Big)^{\frac{1}{p}},$$

and the proof goes the same way.

(4) In [ALWW] the authors considered more general type of nests —called Modeled on Subsets (MOS-nests). These include the previous examples and also nests with various multiplicities. Their main Proposition says that for $p = 1$, T_n converges in the strong operator topology of $B(L_1)$. From there they easily get that the similarity transformations act absolutely continuous and that the change of multiplicity that takes place for $p = 2$ does not occur. However, if $1 < p < \infty$, T_n does not converge in the strong operator topology of $B(L_p)$; and although we were able to handle the absolutely continuous part we do not know how to apply our method to the multiplicity case.

(5) Finally, we just point out that when studying the similarity theory for continuous nests in general Banach spaces one has to restrict the type of the nests (like MOS-nests as above). Because it is possible to have a continuous nests such that any two elements of the nest are non-isomorphic (see [AF]).

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