FACTORIZATION AND REFLEXIVITY ON FOCK SPACES

Alvaro Arias and Gelu Popescu

The framework of the paper is that of the full Fock space $\mathcal{F}^2(\mathcal{H}_n)$ and the Banach algebra F^∞ which can be viewed as non-commutative analogues of the Hardy spaces H^2 and H^∞ respectively.

An inner-outer factorization for any element in $\mathcal{F}^2(\mathcal{H}_n)$ as well as characterization of invertible elements in F^∞ are obtained. We also give a complete characterization of invariant subspaces for the left creation operators $S_1, \cdots, S_n$ of $\mathcal{F}^2(\mathcal{H}_n)$. This enables us to show that every weakly (strongly) closed unital subalgebra of $\{\varphi(S_1, \cdots, S_n) : \varphi \in F^\infty\}$ is reflexive, extending in this way the classical result of Sarason [S]. Some properties of inner and outer functions and many examples are also considered.

0. INTRODUCTION. Let H^p, $1 \le p \le \infty$ be the classical Hardy spaces on the disk. H^2 is a Hilbert space with orthonormal basis $\{z^n\}_{n=0}^\infty$ and it is well known that if $\varphi \in H^\infty$, then $\|\varphi\|_\infty = \|M_\varphi\|$, where $M_\varphi : H^2 \to H^2$ is the multiplication operator. Hence,

$$\|\varphi\|_\infty = \sup\,\{\|\varphi p\|_2 : \|p\|_2 \le 1, \text{ and } p \text{ is a polynomial on } z\}. \tag{1}$$

The set of polynomials on z, $\mathcal{P}(z)$, determine the Hardy spaces: H^2 is the closure of $\mathcal{P}(z)$ in the Hilbert space with orthonormal basis $\{z^n\}_{n=0}^\infty$. Once we have H^2, H^∞ consists of all $\varphi \in H^2$ such that (1) is finite.

In [Po3], [Po4] the second author introduced and studied a "non-commutative" analogue of H^∞. Let $\mathcal{P}$ be the set of polynomials in n non-commutative indeterminates $e_1, \cdots, e_n$. To stress the non-commutativity of the product we use tensor notation; (i.e., $e_1 \otimes e_2$ represents the product of e_1 and e_2 and is different from $e_2 \otimes e_1$). A typical element of $\mathcal{P}$ looks like

$$p = a_0 + \sum_{k=1}^{m} \sum_{1 \le i_1, \cdots, i_k \le n} a_{i_1 \cdots i_k} e_{i_1} \otimes \cdots \otimes e_{i_k}, \tag{2}$$

where $m \in \mathbf{N}$ and $a_0, a_{i_1 \cdots i_k} \in \mathbf{C}$. Let $\mathcal{F}^2(\mathcal{H}_n)$ be the Hilbert space having the monomials as an orthonormal basis (i.e., 1 and the elements of the form $e_{i_1} \otimes \cdots \otimes e_{i_k}$). This will be viewed as analogue to H^2; $\mathcal{P}$ is dense there.

Define F^∞ as the set of all $g \in \mathcal{F}^2(\mathcal{H}_n)$ such that

$$\|g\|_\infty = \sup \{\|g \otimes p\|_2 : p \in \mathcal{P}, \|p\|_2 \leq 1\} < \infty, \tag{3}$$

where $\|\cdot\|_2 = \|\cdot\|_{\mathcal{F}^2(\mathcal{H}_n)}$.

It is easy to prove (see [Po3]) that $(F^\infty, \|\cdot\|_\infty)$ is a non-commutative Banach algebra (if $f, g \in F^\infty$, then $f \otimes g \in F^\infty$ and $\|f \otimes g\|_\infty \leq \|f\|_\infty \|g\|_\infty$) which can be viewed as a non-commutative analogue of the Hardy space H^∞; when $n = 1$ they coincide.

F^∞ shares many properties with H^∞. For instance, the second author (see [Po3]) extended the classical von Neumann's inequality [vN] to polynomials in F^∞. He proved that if $T_1, \cdots, T_n \in B(\mathcal{H})$, for some Hilbert space $\mathcal{H}$, are such that the operator matrix $[T_1 \cdots T_n]$ is a contraction (i.e., $\|\sum_{i=1}^n T_i T_i^*\| \leq 1$) then for every $p \in \mathcal{P}$,

$$\|p(T_1, \cdots, T_n)\| \leq \|p\|_\infty,$$

where $p(T_1, \cdots, T_n)$ is obtained from (2) by replacing the e_i's by T_i's.

There are also many analogies with the invariant subspaces of H^2, inner and outer functions in H^∞, Toeplitz operators, etc. (see [Po2], [Po4]). We will give precise definitions for many of these facts below.

For $i \leq n$ define $S_i : \mathcal{F}^2(\mathcal{H}_n) \to \mathcal{F}^2(\mathcal{H}_n)$ by $S_i \psi = e_i \otimes \psi$. These are unilateral shifts with orthogonal final spaces and are analogue to $S = M_z : H^2 \to H^2$, multiplication by z, the unilateral shift on H^2.

For $\varphi \in F^\infty$ define $\varphi(S_1, \cdots, S_n) : \mathcal{F}^2(\mathcal{H}_n) \to \mathcal{F}^2(\mathcal{H}_n)$ by $\varphi(S_1, \cdots, S_n)\psi = \varphi \otimes \psi$ for every $\psi \in \mathcal{F}^2(\mathcal{H}_n)$. Following the classical case the second author (see [Po4]) defined:

1. $\varphi \in F^\infty$ is *inner* iff $\varphi(S_1, \cdots, S_n)$ is an isometry on $\mathcal{F}^2(\mathcal{H}_n)$, and
2. $\varphi \in F^\infty$ is *outer* iff $\varphi \otimes \mathcal{F}^2(\mathcal{H}_n)$ is dense in $\mathcal{F}^2(\mathcal{H}_n)$.

In this paper we define

3. $\psi \in \mathcal{F}^2(\mathcal{H}_n)$ is *outer* iff $\psi \otimes \mathcal{P}$ is dense in $\mathcal{F}^2(\mathcal{H}_n)$.

In Section 1 we will set the notation, terminology and prove a few elementary facts about $\mathcal{F}^2(\mathcal{H}_n)$, F^∞ and inner and outer functions. In Section 2 we will study some factorization properties of inner and outer functions and will complete the characterization of the invariant subspaces for $S_1, \cdots, S_n$ started in [Po1]. In Section 3 we will provide many examples of inner and outer functions. We will see that the theory of inner functions is more interesting in the non-commutative case than in the commutative one. In Section 4 we will prove that every weakly (strongly) closed unital subalgebra $\mathcal{A}$ of F^∞ is reflexive; i.e., Alg Lat $\mathcal{A} = \mathcal{A}$. The proof of this uses most of the examples of inner functions presented in Section 3. We will finish in Section 5 with some open problems.

1. PRELIMINARIES. In this section we will set the notation, terminology and prove a few elementary facts.

$\mathcal{F}^2(\mathcal{H}_n)$ is called the full Fock space on the n-dimensional Hilbert space $\mathcal{H}_n$ with orthonormal basis $(e_1, \cdots, e_n)$. It is usually represented by

$$\mathcal{F}^2(\mathcal{H}_n) = \mathbf{C}1 \oplus \bigoplus_{m \geq 1} \mathcal{H}_n^{\otimes m}.$$

Let $\Lambda = \{1, 2, \cdots, n\}$ be fixed throughout the paper. For every $k \geq 1$, let $F(k, \Lambda)$ be the set of all functions from $\{1, 2, \cdots, k\}$ to Λ, and let

$$\mathcal{F} = \bigcup_{k=0}^{\infty} F(k, \Lambda),$$

where $F(0, \Lambda)$ stands for $\{0\}$. We use $\mathcal{F}$ to describe $\mathcal{F}^2(\mathcal{H}_n)$ and simplify the notation. If $f \in F(k, \Lambda)$, let

$$e_f = e_{f(1)} \otimes e_{f(2)} \otimes \cdots \otimes e_{f(k)}, \quad \text{and for} \quad k = 0, \quad e_0 = 1.$$

Then $\mathcal{F}^2(\mathcal{H}_n)$ is the Hilbert space with orthonormal basis $\{e_f : f \in \mathcal{F}\}$.

Moreover, we use $\mathcal{F}$ to describe finite products. If $T_1, T_2, \cdots, T_n \in B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$, and $f \in F(k, \Lambda)$ we denote

$$T_f = T_{f(1)} T_{f(2)} \cdots T_{f(k)}, \qquad T_0 = I. \tag{4}$$

We will use the following elementary results.

LEMMA 1.1. *Let $\varphi, \psi \in F^\infty$. If $\varphi \otimes \psi = 0$, then $\varphi = 0$ or $\psi = 0$.*

Proof. Let $\varphi = \sum_{f \in \mathcal{F}} a_f e_f$ and $\psi = \sum_{g \in \mathcal{F}} b_g e_g$ be non-zero elements. Let k is the smallest integer such that for some $\bar{f} \in F(k, \Lambda)$, $a_{\bar{f}} \neq 0$, and ℓ the smallest integer such that for some $\bar{g} \in F(\ell, \Lambda)$, $b_{\bar{g}} \neq 0$.

We have that $\varphi \otimes \psi = \sum_{h \in \mathcal{F}} \left[\sum_{e_f \otimes e_g = e_h} a_f b_g\right] e_h$. Let $\bar{h}$ be such that $e_{\bar{h}} = e_{\bar{f}} \otimes e_{\bar{g}}$. It is easy to see that

$$\langle \varphi \otimes \psi, e_{\bar{h}} \rangle = \sum_{e_f \otimes e_g = e_{\bar{h}}} a_f b_g = a_{\bar{f}} b_{\bar{g}} \neq 0.$$

Hence, $\varphi \otimes \psi \neq 0$. ∎

COROLLARY 1.2. *Suppose that $\varphi \in F^\infty$, $\varphi \neq 0$ and that $\varphi = \varphi \otimes \psi$. Then $\psi = 1$.*

Proof. Let φ and ψ satisfy $\varphi = \varphi \otimes \psi$. Then $\varphi \otimes (1 - \psi) = 0$. Since $\varphi \neq 0$, we get that $1 - \psi = 0$. ∎

Recall that $\varphi \in F^\infty$ is inner iff the map $\psi \to \varphi \otimes \psi$ is an isometry on $\mathcal{F}^2(\mathcal{H}_n)$ and that $\psi \in F^\infty$ is outer iff $\psi \otimes \mathcal{F}^2(\mathcal{H}_n)$ is dense in $\mathcal{F}^2(\mathcal{H}_n)$. It is immediate from the definitions that

PROPOSITION 1.3. *φ is inner iff $\{\varphi \otimes e_f : f \in \mathcal{F}\}$ is an orthonormal set in $\mathcal{F}^2(\mathcal{H}_n)$.*

PROPOSITION 1.4 *ψ is outer iff there exist $h_n \in \mathcal{F}^2(\mathcal{H}_n)$ (or $p_n \in \mathcal{P}$) such that $\psi \otimes h_n \to e_0$ ($\psi \otimes p_n \to e_0$) in $\mathcal{F}^2(\mathcal{H}_n)$.*

We will also use

PROPOSITION 1.5. *Let φ be inner. The map $\psi \to \varphi \otimes \psi$ is an isometry on F^∞. Moreover, if $\phi = \varphi \otimes \psi$, then $\psi \in F^\infty$ if and only if $\phi \in F^\infty$ and $\|\psi\|_\infty = \|\phi\|_\infty$.*

Proof. Let φ be inner and $\psi \in F^\infty$. For every $p \in \mathcal{P}$ one has

$$\|(\varphi \otimes \psi) \otimes p\|_2 = \|\varphi \otimes (\psi \otimes p)\|_2 = \|\psi \otimes p\|_2.$$

It follows from (3) that $\|\varphi \otimes \psi\|_\infty = \|\psi\|_\infty$.

Suppose now that $\phi = \varphi \otimes \psi$. It is clear that $\sup_{p \in (\mathcal{P})_1} \|\phi \otimes p\|_2$ is finite if and only if $\sup_{p \in (\mathcal{P})_1} \|\psi \otimes p\|_2$ is finite too. ∎

PROPOSITION 1.6. *$\varphi \in F^\infty$ is inner iff $\|\varphi\|_2 = \|\varphi\|_\infty = 1$.*

Proof. Suppose that $\varphi \in F^\infty$ is inner. Then $\|\varphi\|_\infty \geq \|\varphi \otimes e_0\|_2 = \|\varphi\|_2 = 1$. On the other hand, if $p = \sum_f a_f e_f \in \mathcal{P}$, we have that $\|\varphi \otimes p\|_2^2 = \sum_f |a_f|^2 \|\varphi \otimes e_f\|_2^2 = \sum_f |a_f|^2 = \|p\|_2^2$. Therefore $\|\varphi\|_\infty \leq 1$.

Conversely, suppose that $\|\varphi\|_\infty = \|\varphi\|_2 = 1$. Then for every $f \in \mathcal{F}$, $1 = \|\varphi\|_2 = \|\varphi \otimes e_f\|_2$. It is well known that if $x, y \in \ell_2$ satisfy $\|x\|_2 = \|y\|_2 = 1$ and $\|x \pm y\|_2 \leq \sqrt{2}$, then x and y are orthogonal. Let $f, g \in \mathcal{F}$, $f \neq g$. Since $\|\varphi\|_\infty = 1$, $\|\varphi \otimes e_f \pm \varphi \otimes e_g\|_2 = \|\varphi \otimes (e_f \pm e_g)\|_2 \leq \|e_f \pm e_g\|_2 = \sqrt{2}$. Therefore, $\varphi \otimes e_f \perp \varphi \otimes e_g$. ∎

Recall that for $i \leq n$ we defined $S_i : \mathcal{F}^2(\mathcal{H}_n) \to \mathcal{F}^2(\mathcal{H}_n)$ by $S_i\psi = e_i \otimes \psi$. $\mathcal{S} = \{S_i\}_{i \leq n}$ is a sequence of unilateral shifts on $\mathcal{F}^2(H_n)$ with orthogonal final spaces.

A closed subspace $\mathcal{M} \subset \mathcal{F}^2(H_n)$ is *invariant* for $\mathcal{S}$ if $S_i\mathcal{M} \subset \mathcal{M}$ for $i \leq n$. The second author (see [Po2]) defined the *wandering subspace of* $\mathcal{S}|_\mathcal{M}$ by $\mathcal{L} = \mathcal{M} \ominus (\bigoplus_{i \leq n} S_i\mathcal{M})$, and used it to obtain a Wold decomposition for $\mathcal{M}$. Specifically, he proved that

$$f, g \in \mathcal{F},\ f \neq g \Longrightarrow S_f\mathcal{L} \perp S_g\mathcal{L}, \tag{5}$$

(see (4) for the notation of S_f), and

$$\mathcal{M} = \bigoplus_{f \in \mathcal{F}} S_f\mathcal{L}. \tag{6}$$

More generally, a sequence $\mathcal{V} = \{V_\lambda\}_{\lambda\in\Lambda}$ of unilateral shifts on a Hilbert space $\mathcal{H}$ with orthogonal final spaces is a Λ-*orthogonal shift* if the operator matrix $[V_1 V_2 \cdots V_n]$ is non-unitary; i.e., $\mathcal{L} = \mathcal{H} \ominus \left(\bigoplus_{\lambda\in\Lambda} V_\lambda \mathcal{H}\right) \neq \{0\}$. $\mathcal{L}$ is called the *wandering subspace* of $\mathcal{V}$. The formulas (5) and (6) are now

$$f, g \in \mathcal{F},\ f \neq g \Longrightarrow V_f\mathcal{L} \perp V_g\mathcal{L}, \tag{5'}$$

$$\mathcal{H} = \bigoplus_{f\in\mathcal{F}} V_f\mathcal{L}. \tag{6'}$$

Moreover, the dimension of $\mathcal{L}$, called the multiplicity of the shift, determines up to unitary equivalence the Λ-orthogonal shift.

Let $\Theta : \mathcal{F}^2(\mathcal{H}_n) \to \mathcal{F}^2(\mathcal{H}_n)$ be the flip operator defined by $\Theta\varphi = \tilde{\varphi}$ where

$$e_f = e_{f(1)} \otimes \cdots \otimes e_{f(k)}, \qquad \Theta e_f = \tilde{e}_f = e_{f(k)} \otimes e_{f(k-1)} \otimes \cdots \otimes e_{f(1)}.$$

It is easy to check that Θ is unitary and that $\Theta = \Theta^* = \Theta^{-1}$. We use Θ to describe right multiplication on $\mathcal{F}^2(\mathcal{H}_n)$.

LEMMA 1.7. *Let $\psi \in \mathcal{F}^2(\mathcal{H}_n)$ and suppose that*

$$\sup\{\|p \otimes \psi\|_2 \ :\ p \in \mathcal{P},\ \|p\|_2 \leq 1\} < \infty.$$

Then $\tilde{\psi} \in F^\infty$ and the supremum is equal to $\|\tilde{\psi}\|_\infty$.

Proof. Since Θ is an isometry on $\mathcal{F}^2(\mathcal{H}_n)$ and $\Theta(p \otimes q) = \Theta(q) \otimes \Theta(p) = \tilde{q} \otimes \tilde{p}$, we have

$$\sup_{p\in(\mathcal{P})_1} \|p \otimes \psi\|_2 = \sup_{p\in(\mathcal{P})_1} \|\tilde{\psi} \otimes \tilde{p}\|_2 = \sup_{p\in(\mathcal{P})_1} \|\tilde{\psi} \otimes p\|_2 = \|\tilde{\psi}\|_\infty. \quad \blacksquare$$

REMARK. Θ is unbounded in F^∞. Example 6 of Section 3 tells us that if $p(e_1)$ is a polynomial in e_1, then $\|e_2 \otimes p(e_1)\|_\infty = \|p(e_1)\|_\infty$ but $\|\Theta(e_2 \otimes p(e_1))\|_\infty = \|p(e_1) \otimes e_2\|_\infty = \|p(e_1)\|_2$.

2. FACTORIZATION RESULTS.

In this section we will prove:

THEOREM 2.1. *If $\psi \in F^2(H_n)$, $\psi \neq 0$, then there exist φ inner and g outer functions such that $\psi = \varphi \otimes g$. Moreover, the factorization is essentially unique and $\psi \in F^\infty$ if and only if $g \in F^\infty$ and $\|\psi\|_\infty = \|g\|_\infty$.*

And together with [Po3, Corollary 3.5],

THEOREM 2.2. *Let* $\Phi : F^\infty \to B(\mathcal{F}^2(H_n))$ *be defined by* $\Phi(\varphi) = \varphi(S_1, \cdots, S_n)$. *Then:*

(i) Φ *is an algebra homomorphism.*

(ii) $\|\Phi\psi\| = \|\psi\|_\infty$, *for any* $\psi \in F^\infty$.

(iii) ψ *is invertible in* F^∞ *if and only if* $\Phi\psi$ *is invertible in* $B(\mathcal{F}^2(H_n))$.

To prove Theorem 2.1 we will first complete the characterization of invariant subspaces for $S_1, \cdots, S_n$ started in [Po1] for Λ-orthogonal shifts of arbitrary multiplicity.

We say that the inner functions φ_1, φ_2 are *orthogonal* if $\varphi_1 \otimes \mathcal{F}^2(\mathcal{H}_n) \perp \varphi_2 \otimes \mathcal{F}^2(\mathcal{H}_n)$.

THEOREM 2.3. *If* $\mathcal{M} \subset \mathcal{F}^2(H_n)$ *is invariant for each* $S_1, \cdots, S_n$ *then there exists a sequence* $\{\varphi\}_{j\in J}$ *of orthogonal inner functions such that*

$$\mathcal{M} = \bigoplus_{j\in J} \mathcal{F}^2(H_n) \otimes \tilde{\varphi}_j.$$

Moreover, this representation is essentially unique.

Proof. Let $\mathcal{M}$ be a nontrivial closed invariant subspace for $S_1, \cdots, S_n$. According to (6)

$$\mathcal{M} = \bigoplus_{f\in\mathcal{F}} S_f\mathcal{L}, \tag{7}$$

where $\mathcal{L} = \mathcal{M} \ominus (S_1\mathcal{M} \oplus \cdots \oplus S_n\mathcal{M})$ is the wandering subspace for $S_1|_\mathcal{M}, \cdots, S_n|_\mathcal{M}$.

Let $\{\tilde{\varphi}_j\}_{j\in J}$ be an orthonormal basis in $\mathcal{L}$. According to (5), for each $j \in J$

$$S_f\tilde{\varphi}_j \perp S_g\tilde{\varphi}_j, \quad \text{for any} \quad f, g \in \mathcal{F}, f \neq g. \tag{8}$$

By Proposition 1.3 we infer that φ_j is an inner function. On the other hand the relations (7) and (8) imply the following orthogonal decomposition

$$\mathcal{M} = \bigoplus_{j\in J} \mathcal{F}^2(H_n) \otimes \tilde{\varphi}_j.$$

Suppose that $\mathcal{M} = \bigoplus_{i\in I}[\mathcal{F}^2(\mathcal{H}_n) \otimes \tilde{\psi}_i]$ for some orthogonal inner functions ψ_i. Let $\mathcal{L}' = \vee_{i\in I}\tilde{\psi}_i$. It is easy to see that

$$\mathcal{M} = \bigoplus_{f\in S_f} S_f\mathcal{L}'.$$

Hence $\mathcal{L}' = \mathcal{M} \ominus (S_1\mathcal{M} \oplus \cdots \oplus S_n\mathcal{M})$ is the wandering subspace for $S_1|_\mathcal{M}, \cdots, S_n|_\mathcal{M}$ and $\mathcal{L} = \mathcal{L}'$. Since $\{\tilde{\varphi}_j\}_{j\in J}$ and $\{\tilde{\psi}_i\}_{i\in I}$ are orthogonormal basis in $\mathcal{L}$ we infer that they have the same cardinality and that there exists a unitary operator $V \in B(\mathcal{L})$ such that $V\tilde{\varphi}_j = \tilde{\psi}_j$, $j \in J$. ■

COROLLARY 2.4. *If φ_1, φ_2 are inner functions such that $\varphi_1 \otimes \mathcal{F}^2(\mathcal{H}_n) = \varphi_2 \otimes \mathcal{F}^2(\mathcal{H}_n)$, then there exists $\alpha \in \mathbf{C}$, $|\alpha| = 1$ such that $\varphi_1 = \alpha\varphi_2$.*

Proof. Let $\mathcal{M} = \mathcal{F}^2(\mathcal{H}_n) \otimes \tilde{\varphi}_1 = \mathcal{F}^2(\mathcal{H}_n) \otimes \tilde{\varphi}_2$. It is clear that $\mathcal{M}$ is invariant for $S_1, \cdots, S_n$. It follows from the proof of Theorem 2.3 that $[\tilde{\varphi}_1] = [\tilde{\varphi}_2]$. Since $\tilde{\varphi}_1$, $\tilde{\varphi}_2$ are normalized in $\mathcal{F}^2(\mathcal{H}_n)$ there exists $\alpha \in \mathbf{C}$, $|\alpha| = 1$ such that $\tilde{\varphi}_1 = \alpha\tilde{\varphi}_2$. ■

The next lemma will show that all the cyclic invariant subspaces of $\mathcal{F}^2(\mathcal{H}_n)$ are of the form $\mathcal{F}^2(\mathcal{H}_n) \otimes \tilde{\varphi}$ for some inner function φ.

LEMMA 2.5. *Let $\mathcal{V} = \{V_1, \cdots, V_n\}$ be a Λ-orthogonal shift of arbitrary multiplicity acting on the Hilbert space $\mathcal{K}$. If $\mathcal{M} \subset \mathcal{K}$ is a cyclic invariant subspace of $\mathcal{V}$, that is*

$$\mathcal{M} = \bigvee_{f \in \mathcal{F}} V_f \psi, \quad \text{for some } \psi \in \mathcal{K},\ \psi \neq 0$$

then $\mathcal{V}_{|\mathcal{M}} := \{V_1|_{\mathcal{M}}, \cdots, V_n|_{\mathcal{M}}\}$ is unitarily equivalent to the Λ-orthogonal shift of multiplicity 1.

Proof. Let $\psi \in \mathcal{K}$, $\psi \neq 0$ be a fixed element in $\mathcal{K}$ and let

$$\mathcal{M} = \bigvee_{f \in \mathcal{F}} V_f \psi.$$

Since $\mathcal{V}$ is a Λ-orthogonal shift in $\mathcal{K}$ it follows from (5') and (6') that $\mathcal{K} = \bigoplus_{f \in \mathcal{F}} V_f \mathcal{L}$, where $\mathcal{L} = \mathcal{K} \ominus (V_1\mathcal{K} \oplus \cdots \oplus \mathcal{V}_k\mathcal{K})$ is the wandering subspace of $\mathcal{V}$.

Since $\mathcal{V}|_{\mathcal{M}}$ is also a Λ-orthogonal shift let us denote by $\mathcal{L}_0$ its wandering subspace. Therefore, $\mathcal{L}_0 \subset \mathcal{M}$ and

$$\mathcal{M} = \bigoplus_{f \in \mathcal{F}} V_f \mathcal{L}_0, \quad \text{where} \quad \mathcal{L}_0 = \mathcal{M} \ominus (V_1\mathcal{M} \oplus \cdots \oplus V_n\mathcal{M}). \tag{9}$$

Consider $w_0 = P_{\mathcal{L}_0}\psi$, where $P_{\mathcal{L}_0}$ is the orthogonal projection from $\mathcal{K}$ onto $\mathcal{L}_0$. Notice that according to (9) it follows that $w_0 \neq 0$. Let $\ell_0 \in \mathcal{L}_0$ be such that $\ell_0 \perp w_0$. Since $\mathcal{L}_0$ is wandering subspace for $\mathcal{V}|_{\mathcal{M}}$ we have

$$\ell_0 \perp V_f\psi, \quad \text{for any } f \in F(k, \Lambda),\ k \geq 1.$$

On the other hand

$$\langle \ell_0, \psi \rangle = \langle \ell_0,\ P_{\mathcal{L}_0}\psi \rangle = \langle \ell_0, w_0 \rangle = 0.$$

Therefore, $\ell_0 \perp V_f\psi$, for any $f \in \mathcal{F}$. Since $\ell_0 \in \mathcal{M} = \bigvee_{f \in \mathcal{F}} V_f\psi$ we infer that $\ell_0 = 0$. Thus, $\dim \mathcal{L}_0 = 1$ and according to [Po3, Theorem 1.2] it follows that $\mathcal{V}|_{\mathcal{M}}$ is unitarily equivalent to the Λ-orthogonal shift of multiplicity one $\mathcal{S} = \{S_1, \cdots, S_n\}$. The proof is complete. ■

COROLLARY 2.6. *If* $\psi \in \mathcal{F}^2(H_n)$, $\psi \neq 0$ *then there exists an inner function* $\varphi \in F^\infty$ *such that*

$$clos[\mathcal{P} \otimes \psi] = F^2(H_n) \otimes \tilde{\varphi}.$$

Moreover, the representation is essentially unique.

Proof. Let $\mathcal{M} = \text{clos}[\mathcal{P} \otimes \psi]$. It follows from the previous lemma that $\mathcal{L} = \mathcal{M} \ominus (S_1\mathcal{M} \oplus \cdots \oplus S_n\mathcal{M})$, the wandering subspace for $S_1|_{\mathcal{M}}, \cdots, S_n|_{\mathcal{M}}$, has dimension 1. Using Proposition 2.3 we get that $\mathcal{M} = \mathcal{F}^2(\mathcal{H}_n) \otimes \tilde{\varphi}$ for some inner function φ. The "uniqueness" is a consequence of Corollary 2.4. ■

Proof of Theorem 2.1. Let $\mathcal{M} = \text{clos}\ [\mathcal{P} \otimes \tilde{\psi}]$. Then $\mathcal{M}$ is a nontrivial closed invariant subspace for each $S_1, \cdots, S_n$. By Corollary 2.6 it is of the form $\mathcal{M} = F^2(H_n) \otimes \tilde{\varphi}$, where $\varphi \in F^\infty$ is inner. Since $\tilde{\psi} \in \mathcal{M}$, there must exist $\tilde{g} \in F^2(H_n)$ such that

$$\tilde{\psi} = \tilde{g} \otimes \tilde{\varphi} \quad \text{and then} \quad \psi = \varphi \otimes g.$$

Since $\tilde{\varphi} \in \mathcal{M}$, there exist $p_n \in \mathcal{P}$ such that

$$\tilde{\varphi} = \lim_{n\to\infty} p_n \otimes \tilde{\psi} = \lim_{n\to\infty} (p_n \otimes \tilde{g}) \otimes \tilde{\varphi} = \left[\lim_{n\to\infty} p_n \otimes \tilde{g}\right] \otimes \tilde{\varphi}.$$

We use that φ is inner for the last equality. By Corollary 1.2, $\lim_{n\to\infty} p_n \otimes \tilde{g} = e_0$. Hence $\lim_{n\to\infty} g \otimes \tilde{p}_n = e_0$, and using Proposition 1.4 we conclude that g is outer. It follows from Proposition 1.5 that $g \in F^\infty$ iff $\psi \in F^\infty$.

Suppose now that $\psi = \varphi_1 \otimes g_1 = \varphi_2 \otimes g_2$ for some $\varphi_1,\ \varphi_2$ inner and $g_1,\ g_2$ outer. Then we have that $\varphi_1 \otimes \mathcal{F}^2(\mathcal{H}_n) = \varphi_2 \otimes \mathcal{F}^2(\mathcal{H}_n)$, and using Corollary 2.4 we get that $\varphi_1 = \alpha\varphi_2$ for some $\alpha \in \mathbf{C}$, $|\alpha| = 1$. Hence $0 = \varphi_1 \otimes g_1 - \varphi_2 \otimes g_2 = \varphi_2 \otimes (\alpha g_1 - g_2)$. By Lemma 1.1, $\alpha g_1 = g_2$. ■

As in the classical case, there is a factorization result for inner functions.

COROLLARY 2.7. *Let* $\varphi_1,\ \varphi_2$ *be inner functions.* $\varphi_1 \otimes \mathcal{F}^2(\mathcal{H}_n) \subset \varphi_2 \otimes \mathcal{F}^2(\mathcal{H}_n)$ *if and only if there exists* φ_3 *inner such that* $\varphi_1 = \varphi_2 \otimes \varphi_3$.

Proof. Suppose that $\varphi_1 \otimes \mathcal{F}^2(\mathcal{H}_n) \subset \varphi_2 \otimes \mathcal{F}^2(\mathcal{H}_n)$. Then $\varphi_1 = \varphi_2 \otimes \psi$ for some $\psi \in \mathcal{F}^2(\mathcal{H}_n)$. Let $\psi = \psi_i \otimes \psi_e$ be the inner-outer factorization of ψ. Therefore, $\varphi_1 = (\varphi_2 \otimes \psi_i) \otimes \psi_e$. By the uniqueness of the factorization of φ_1 we deduce that $\psi_e = 1$. The converse is clear. ■

We will finish the details of Theorem 2.2 now. Parts (i) and (ii) appear in [Po3]. The third part is included in the proof of the next theorem.

THEOREM 2.8. $\varphi \in F^\infty$ *is invertible if and only if* φ *is outer and there exists* $\delta > 0$ *such that* $\|\varphi \otimes p\|_2 \geq \delta \|p\|_2$, *for any* $p \in \mathcal{P}$.

Proof. Let $\varphi \in F^\infty$ be invertible and find $\psi \in F^\infty$ such that $\varphi \otimes \psi = \psi \otimes \varphi = e_0$. Hence we infer that

$$\varphi(S_1, \cdots, S_n)\psi(S_1, \cdots, S_n) = \psi(S_1, \cdots, S_n)\varphi(S_1, \cdots, S_n) = I_{F^2(H_n)}.$$

Therefore the operator $\varphi(S_1, \cdots, S_n)$ is invertible which implies:

$$\text{range } \varphi(S_1, \cdots, S_n) = F^2(H_n), \qquad \text{and} \tag{10}$$

$$\varphi(S_1, \cdots, S_n) \quad \text{is bounded below.} \tag{11}$$

The relation (10) shows that φ is outer and (11) implies that there exists $\delta > 0$ such that

$$\|\varphi \otimes p\|_2 \geq \delta \|p\|_2, \quad \text{for any } p \in \mathcal{P}. \tag{12}$$

Conversely, assume $\varphi \in F^\infty$ is outer such that (12) holds. This implies that $\varphi(S_1, \cdots, S_n)$ is invertible in $B(F^2(H_n))$. Let $T \in B(F^2(H_n))$ be such that

$$T\varphi(S_1, \cdots, S_n) = \varphi(S_1, \cdots, S_n)T = I_{F^2(H_n)}.$$

Then for any $h \in F^2(H_n)$ we have

$$T(\varphi \otimes h) = \varphi \otimes Th = h. \tag{13}$$

Let $h = e_0$ and $\psi = Te_0$. Then (13) gives us

$$\varphi \otimes \psi = e_0 \quad \text{and} \quad T(\varphi) = e_0. \tag{14}$$

We clearly have

$$\varphi = e_0 \otimes \varphi = (\varphi \otimes \psi) \otimes \varphi = \varphi \otimes (\psi \otimes \varphi).$$

Then according to (14) and (13) we have $e_0 = T(\varphi) = T(\varphi \otimes (\psi \otimes \varphi)) = \psi \otimes \varphi$. Therefore,

$$\varphi \otimes \psi = \psi \otimes \varphi = e_0.$$

We claim that $\psi \in F^\infty$. Let $p \in \mathcal{P}$. It is clear that

$$p = e_0 \otimes p = (\varphi \otimes \psi) \otimes p = \varphi \otimes (\psi \otimes p).$$

Therefore, by (13)

$$T(p) = T(\varphi \otimes (\psi \otimes p)) = \psi \otimes p.$$

Hence,

$$\|\psi\|_\infty = \sup_{p \in (\mathcal{P})_1} \|\psi \otimes p\|_2 = \sup_{p \in (\mathcal{P})_1} \|Tp\|_2 = \|T\| < \infty.$$

The proof is complete. ■

3. EXAMPLES OF INNER AND OUTER FUNCTIONS. In this section we will present many examples of inner and outer functions in F^∞. Recall that $\varphi \in F^\infty$ is inner iff the map $h \to \varphi \otimes h$ is an isometry on $\mathcal{F}^2(\mathcal{H}_n)$, (equivalently, $h \to h \otimes \tilde{\varphi}$ is an isometry on $\mathcal{F}^2(\mathcal{H}_n)$), and $\varphi \in F^\infty$ is outer iff there exists $p_n \in \mathcal{F}^2(\mathcal{H}_n)$ such that $\varphi \otimes p_n \to e_0$ in $\mathcal{F}^2(H_n)$.

The first observation is that any inner (respectively outer) function in H^∞ becomes inner (respectively outer) in F^∞ if we substitute z by e_f. We call these examples "inherited".

EXAMPLE 1. *Inherited inner and outer functions.*

For any $f \in \mathcal{F}$, $f \neq 0$ let $V_f : H^2 \to \mathcal{F}^2(H_n)$ be the isometry defined by $V_f z^k = e_f^k$. We denote $V_f \varphi = \varphi_f$.

PROPOSITION 3.1. $V_f : H^\infty \to F^\infty$ *is an isometry. Moreover, if* $\varphi \in H^\infty$ *is inner then* φ_f *is inner in* F^∞*, and if* $\varphi \in H^\infty$ *is outer then* φ_f *is outer in* F^∞.

Proof. Let $\mathcal{F}_f^2 = [e_0, e_f, e_f^2, \cdots]$. Any $h \in \mathcal{F}$ satisfies $e_h = e_f^k \otimes e_g$ for some $k \in \mathbf{N}_0$ and $g \in \mathcal{F}$ that "does-not-start" from f (i.e., $S_f^* e_g = 0$, see (4) for the notation of S_f). We decompose $\mathcal{F}^2(\mathcal{H}_n)$ as

$$\mathcal{F}^2(\mathcal{H}_n) = \bigoplus_{\substack{g \in \mathcal{F} \\ S_f^* e_g = 0}} [\mathcal{F}_f^2 \oplus e_g]. \tag{15}$$

Since $\varphi_f \in \mathcal{F}_f^2$ we have that for any $g \in \mathcal{F}$ that does not start from f,

$$\varphi_f \otimes [\mathcal{F}_f^2 \otimes e_g] \subset \mathcal{F}_f^2 \otimes e_g.$$

Moreover, it is easy to see that

$$\sup_{p \in (\mathcal{F}_f^2 \otimes e_g)_1} \|\varphi_f \otimes p\|_2 = \sup_{p \in (\mathcal{F}_f^2)_1} \|\varphi_f \otimes p\|_2 = \sup_{p \in (H^2)_1} \|\varphi p\|_2 = \|\varphi\|_\infty.$$

Combining this with the fact that φ_f acts diagonally in the decomposition of $\mathcal{F}^2(\mathcal{H}_n)$ given by (15) we conclude that $\|\varphi_f\|_\infty = \|\varphi\|_\infty$.

Suppose that $\varphi \in H^\infty$ is outer. Then we can find a sequence of $p_n \in H^2$ such that $\varphi p_n \to 1$ in H^2. Since V_f is an isometry on H^2 we get that $V_f(\varphi p_n) = \varphi_f \otimes V_f(p_n) \to e_0$ in $\mathcal{F}^2(\mathcal{H}_n)$. By Proposition 1.4 we get that φ_f is outer in F^∞.

Suppose now that $\varphi \in H^\infty$ is inner. From the classical theory we have that $\|\varphi\|_2 = \|\varphi\|_\infty = 1$. Since V_f is an isometry on H^2 and H^∞ we get that $\|\varphi_f\|_2 = \|\varphi_f\|_\infty = 1$. Therefore, by Propsition 1.6, φ_f is inner. ■

EXAMPLE 2. *(Invertible elements) If* $\psi \in F^\infty$ *is invertible, then* ψ *is outer.*

In particular, the following examples are invertible (and hence outer):

(i) For $\varphi_i \in F^\infty$, $\|\varphi_i\|_\infty < 1$ for $i \le k$, let

$$\psi = (e_0 - \varphi_1) \otimes (e_0 - \varphi_2) \otimes \cdots \otimes (e_0 - \varphi_k).$$

(ii) For $\varphi \in F^\infty$, and $\varphi^n = \varphi \otimes \cdots \otimes \varphi$ (n-times), let

$$\psi = \exp \varphi = \sum_{k=0}^{\infty} \frac{\varphi^n}{n!}.$$

(iii) For $\lambda \in \mathbf{C}^n$, $\|\lambda\|_2 < 1$ let $\psi = z_\lambda$, where the z_λ's are defined in Example 8.

We will present examples of inner functions now. The first example is the simplest.

EXAMPLE 3. *(The monomials) For every $f \in \mathcal{F}$, e_f is inner.*

Examples 4 and 6 below appear in [A].

EXAMPLE 4. *For every $k \in \mathbf{N}$ let $X_k =$ span $\{e_f : f \in F(k,\Lambda)\}$ (i.e., the span of the monomials with k letters). Then any $x \in X_k$, $\|x\|_2 = 1$ is inner.*

Proof. The main point is that if $f_1, f_2 \in F(k,\Lambda)$ and $g, h \in \mathcal{F}$ then $e_{f_1} \otimes e_g = e_{f_2} \otimes e_h$ iff $f_1 = f_2$ and $g = h$.

We can easily see that if $x = \sum_{f \in F(k,\Lambda)} a_f e_f \in X_k$ and $g, h \in \mathcal{F}$, $g \ne h$ then $x \otimes e_g$ is orthogonal to $x \otimes e_h$. If we require that $\|x\|_2 = 1$, x is inner. ■

EXAMPLE 5. *Let $e_{f_1}, \cdots, e_{f_k}$ be monomials such that the first letter of all of them are different. (i.e., $k \ne \ell$ implies that $f_k(1) \ne f_\ell(1)$). Then for any $\sum_{i \le k} |a_i|^2 = 1$ we have that $\sum_{i \le k} a_i e_{f_i}$ is inner.*

Proof. The proof is similar to that of Example 4. Since the f_i's start from different letters, for any $g, h \in \mathcal{F}$, $e_{f_i} \otimes e_g \perp e_{f_j} \otimes e_h$ if $i \ne j$.

We can easily see that if $x = \sum_{i \le k} a_i e_{f_i}$ and $g, h \in \mathcal{F}$, $g \ne h$ then $x \otimes e_g$ is orthogonal to $x \otimes e_h$. If we require that $\|x\|_2 = 1$, x is inner. ■

EXAMPLE 6. *Let $\psi \in \mathcal{F}^2(H_{n-1})$, $\|\psi\|_2 = 1$, (i.e., ψ does not have any e_n). Then $\psi \otimes e_n$ is inner.*

Proof. Let $\mathcal{F}_{n-1}$ be the set of words on the letters $e_1, \cdots, e_{n-1}$. Let $f_1, f_2 \in \mathcal{F}_{n-1}$ and $g_1, g_2 \in \mathcal{F}$. The main point is that

$$e_{f_1} \otimes e_n \otimes e_{g_1} = e_{f_2} \otimes e_n \otimes e_{g_2} \quad \text{if and only if} \quad f_1 = f_2 \text{ and } g_1 = g_2.$$

To see this notice that the first time that e_n appears in $e_{f_1} \otimes e_n \otimes e_{g_1}$ is right after e_{f_1}. Since the same is true for f_2 the words must agree before e_n (i.e., $f_1 = f_2$) and after (i.e., $g_1 = g_2$).

It is easy to prove now that if $g_1, g_2 \in \mathcal{F}$, $g_1 \ne g_2$, then $\psi \otimes e_n \otimes e_{g_1} \perp \psi \otimes e_n \otimes e_{g_2}$. The normalization condition guarantees that $\psi \otimes e_n$ is inner. ■

EXAMPLE 7. *(Möbius functions) For any $f \in \mathcal{F}$, and $\mu \in \mathbf{C}$ with $|\mu| < 1$, we have that $\varphi(f,\mu) = (e_f - \mu) \otimes (1 - \bar{\mu} e_f)^{-1}$ is inner.*

These are particular cases for Example 1 above. The products of these elements can be viewed as analogue to the Blashke products. However, $[\varphi(f,\mu) \otimes \mathcal{F}^2(\mathcal{H}_n)]^\perp$ is always infinite codimensional. In fact, since $(1 - \bar{\mu} e_f)^{-1}$ is invertible and $(1 - \bar{\mu} e_f)^{-1}$, $(e_f - \mu)$ commute, we have that $h \in [\varphi(f,\mu) \otimes \mathcal{F}^2(\mathcal{H}_n)]^\perp$ iff for every $\psi \in \mathcal{F}^2(\mathcal{H}_n)$, $\langle (e_f - \mu) \otimes \psi, h \rangle = 0$. Then, looking only at the basic elements we get that

$$h \in [\varphi(f,\mu) \otimes \mathcal{F}^2(\mathcal{H}_n)]^\perp \iff \forall g \in \mathcal{F},\ \langle e_f \otimes e_g, h \rangle = \mu \langle e_g, h \rangle. \tag{16}$$

It is easy to see that

$$h_f(\mu) = \sum_{k=0}^{\infty} \mu^k e_f^k$$

satisfies (16) and thus belongs to $[\varphi(f,\mu) \otimes \mathcal{F}^2(\mathcal{H}_n)]^\perp$.

One can also check that if $g \in \mathcal{F}$ "does-not-start" from f (i.e., $S_f^* e_g = 0$, see (4) for the notation of S_f), then $h_f(\mu) \otimes e_g \in [\varphi(f,\mu) \otimes \mathcal{F}^2(\mathcal{H}_n)]^\perp$. Moreover, the span of these elements is dense in $[\varphi(f,\mu) \otimes \mathcal{F}^2(\mathcal{H}_n)]^\perp$. We leave the details out.

We need the following lemma in the next section.

LEMMA 3.2. *Let $\Omega \subset \{\mu \in \mathbf{C} : |\mu| < 1\}$ be a set with an accumulation point in $\{\mu \in \mathbf{C} : |\mu| < 1\}$. Then $\bigcap\{\varphi(f,\mu) \otimes \mathcal{F}^2(\mathcal{H}_n) : f \in \mathcal{F}, \mu \in \Omega\} = \{0\}$.*

Proof. Let $\psi \in \bigcap\{\varphi(f,\mu) \otimes \mathcal{F}^2(\mathcal{H}_n) : f \in \mathcal{F}, \mu \in \Omega\}$ and fix $f \in \mathcal{F}$. For every $\mu \in \Omega$, we have that $\langle \psi, h_f(\mu) \rangle = 0$. Hence,

$$\langle \psi, e_0 \rangle + \mu \langle \psi, e_f \rangle + \mu^2 \langle \psi, e_f^2 \rangle + \mu^3 \langle \psi, e_f^3 \rangle + \cdots = 0 \qquad \text{for all} \quad \mu \in \Omega.$$

Since the map $\mu \to \sum_{k=0}^{\infty} \mu^k \langle \psi, e_f^k \rangle$ is analytic on μ and the zeros accumulate inside the disk, the map is zero. Hence $\langle \psi, e_0 \rangle = \langle \psi, e_f \rangle = 0$. Since f is arbitrary, we conclude that $\psi = 0$. ■

The following example is more technical and it is not used in the rest of the paper. However, it gives rise to non-trivial inner functions and to an interesting ideal in F^∞ that seems to capture the "non-commutativity" of the product.

EXAMPLE 8. *1-codimensional invariant subspaces.*

Let $\mathcal{M}$ be a 1-codimensional subpace of $\mathcal{F}^2(\mathcal{H}_n)$ invariant for $S_1, \cdots, S_n$. Then $\mathcal{M} = [z]^\perp$ for some $z \in \mathcal{F}^2(\mathcal{H}_n)$ and $[z]$ is invariant for $S_1^*, \cdots, S_n^*$. That is, for every $i \le n$ there exists $\lambda_i \in \mathbf{C}$ such that

$$S_i^* z = \lambda_i z.$$

Assume that $\langle z, e_0\rangle = 1$.

For $f \in F(k,\Lambda)$ let $\lambda_f = \lambda_{f(1)}\lambda_{f(2)}\cdots\lambda_{f(k)}$ and $\lambda_0 = 1$. We claim that for any $f \in \mathcal{F}$

$$\langle z, e_f\rangle = \lambda_f \quad \text{and then} \quad z = \sum_{f\in\mathcal{F}} \lambda_f e_f. \tag{17}$$

To see this notice that for every $i \le n$, $\langle z, e_i\rangle = \langle z, S_i e_0\rangle = \langle S_i^* z, e_0\rangle = \lambda_i$. Similarly, $\langle z, e_i \otimes e_j\rangle = \langle z, S_i S_j e_0\rangle = \langle S_j^* S_i^* z, e_0\rangle = \lambda_i\lambda_j$. Proceeding inductively we get (17). Since the λ_i's determine z and $\mathcal{M}$ we will denote them by z_λ and $\mathcal{M}_\lambda$ from now on.

It is not hard to see that

$$z_\lambda = \sum_{f\in\mathcal{F}} \lambda_f e_f = 1 + \sum_{k=1}^{\infty} (\lambda_1 e_1 + \lambda_2 e_2 \cdots + \lambda_n e_n)^k.$$

Then a necessary and sufficient condition for $z_\lambda \in \mathcal{F}^2(\mathcal{H}_n)$ is that $\|\lambda\|_2 = \sqrt{\sum_{i\le n} |\lambda_i|^2} < 1$. Moreover, by Example 4 and Proposition 1.6, $\|\lambda_1 e_1 + \cdots \lambda_n e_n\|_2 = \|\lambda_1 e_1 + \cdots \lambda_n e_n\|_\infty$ and then $z_\lambda \in F^\infty$.

We have thus proved:

PROPOSITION 3.3. *$\mathcal{M}$ is a 1-codimensional invariant subspace for $S_1, \cdots, S_n$ if and only if $\mathcal{M} = [z_\lambda]^\perp$ for some $\lambda \in \mathbf{C}^n$, $\|\lambda\|_2 < 1$.*

It follows from Proposition 2.3 that $\mathcal{M}_\lambda = \bigoplus_{j\in J}[\mathcal{F}^2(\mathcal{H}_n) \otimes \tilde{\varphi}_j]$ for some orthogonal inner functions φ_j, $j \in J$. Recall that $|J|$, the cardinality of J, is the dimension of the wandering subspace $\mathcal{L}_\lambda = \mathcal{M}_\lambda \ominus [S_1\mathcal{M}_\lambda \oplus \cdots \oplus S_n\mathcal{M}_\lambda]$ for $S_1|_{\mathcal{M}_\lambda}, \cdots, S_n|_{\mathcal{M}_\lambda}$. Moreover, whenever $\tilde{\varphi} \in \mathcal{L}_\lambda$ satisfies $\|\tilde{\varphi}\|_2 = 1$ we have that φ is inner.

Let Q_λ be the orthogonal projection onto $\mathcal{M}_\lambda$ and P_λ the orthogonal projection onto $\mathcal{L}_\lambda$. One can easily check that $P_\lambda = Q_\lambda - \sum_{i\le n} S_i Q_\lambda S_i^*$ and that

$$\begin{aligned} Q_\lambda e_f &= e_f - \frac{\lambda_f}{\|z_\lambda\|_2^2} z_\lambda \quad \text{for any} \quad f \in \mathcal{F}, \\ P_\lambda e_i &= \frac{1}{\|z_\lambda\|_2^2}(e_i - \lambda_i) \otimes z_\lambda \quad \text{for} \quad i \le n, \\ P_\lambda e_0 &= e_0 - \frac{1}{\|z_\lambda\|_2^2} z_\lambda. \end{aligned}$$

If $e_f = e_i \otimes e_g$ for some $i \le n$ and $g \in \mathcal{F}$ we can also check that $P_\lambda e_f = \frac{\lambda_g}{\|z_\lambda\|_2^2}(e_i - \lambda_i) \otimes z_\lambda$.

It is easy to see that (17) implies that $\tilde{z}_\lambda = z_\lambda$, since $\lambda_f = \lambda_{\tilde{f}}$. Then we get

PROPOSITION 3.4. *$\mathcal{L}_\lambda = span\{\tilde{\varphi}_0, \tilde{\varphi}_1, \cdots, \tilde{\varphi}_n\}$ where*

$$\varphi_0 = a_0\big[e_0 - \frac{1}{\|z_\lambda\|_2^2} z_\lambda\big], \quad \text{and} \quad \varphi_i = a_i\big[z_\lambda \otimes (e_i - \lambda_i)\big] \quad i = 1, 2 \cdots, n.$$

The $a_i \in \mathbf{C}$ are chosen so that $\|\varphi_i\|_2 = 1$ for $i = 0, 1, \cdots, n$. Moreover, the φ_i's and any normalized linear combination of them are inner function.

REMARKS. (1) We have that $\mathcal{M}_\lambda = \mathcal{F}^2(\mathcal{H}_n)\otimes\tilde{\varphi}_0+\mathcal{F}^2(\mathcal{H}_n)\otimes\tilde{\varphi}_1+\cdots+\mathcal{F}^2(\mathcal{H}_n)\otimes\tilde{\varphi}_n$. However, the φ_i's are not orthogonal inner functions.

(2) We also have that $\psi \in \mathcal{M}_\lambda$ if and only if $\langle\psi, z_\lambda\rangle = 0$ if and only if there exist $g_i \in \mathcal{F}^2(\mathcal{H}_n)$, $i = 0, 1, \cdots, n$ such that

$$\psi = g_0 \otimes \tilde{\varphi}_0 + g_1 \otimes \tilde{\varphi}_1 + \cdots + g_n \otimes \tilde{\varphi}_n.$$

This can be view as a "weak-factorization" result.

(3) It is easy to compute that $\|z_\lambda\|_2^2 = \frac{1}{1-\|\lambda\|_2^2}$.

If $\mathcal{M}$ is an invariant subspace for $S_1, \cdots, S_n$ and $p \in \mathcal{P}$ we always have (Proposition 2.3) that $p \otimes \mathcal{M} \subset \mathcal{M}$. In general we do not have that $\mathcal{M} \otimes p \subset \mathcal{M}$. However, this is always true for the 1-codimensional invariant subspaces.

LEMMA 3.5. *Let $\mathcal{M}_\lambda = [z_\lambda]^\perp$ for some $\lambda \in \mathbf{C}^n$, $\|\lambda\|_2 < 1$. For any $p \in \mathcal{P}$ we have that $p \otimes \mathcal{M}_\lambda \subset \mathcal{M}_\lambda$ and $\mathcal{M}_\lambda \otimes p \subset \mathcal{M}_\lambda$.*

Proof. The first inclusion is clear. If $\varphi \in \mathcal{M}_\lambda$ and $p \in \mathcal{P}$ then $\langle p \otimes \varphi, z_\lambda\rangle = 0$.

The key point for the other one is that $\varphi \in \mathcal{M}_\lambda$ if and only if $\tilde{\varphi} \in \mathcal{M}_\lambda$. To see this recall that $z_\lambda = \tilde{z}_\lambda$. Then

$$\langle\tilde{\varphi}, z_\lambda\rangle = \langle\varphi, \tilde{z}_\lambda\rangle = \langle\varphi, z_\lambda\rangle.$$

If $p \in \mathcal{P}$ and $\varphi \in \mathcal{M}_\lambda$ we have that

$$\langle\varphi \otimes p, z_\lambda\rangle = \langle\varphi \otimes p, \tilde{z}_\lambda\rangle = \langle\widetilde{\varphi \otimes p}, z_\lambda\rangle = \langle\tilde{p} \otimes \tilde{\varphi}, z_\lambda\rangle = 0. \quad \blacksquare$$

Let $\mathcal{N} = \bigcap_{\|\lambda\|_2<1} \mathcal{M}_\lambda$ and $\mathcal{N}^\infty = \mathcal{N} \bigcap F^\infty$. $\mathcal{N}$ is a "2-sided" invariant subspace and $\mathcal{N}^\infty$ is a two-sided ideal in F^∞.

There are plenty of elements in $\mathcal{N}$. For example $\varphi = e_1 \otimes e_2 - e_2 \otimes e_1 \in \mathcal{N}$. For every $\lambda \in \mathbf{C}^n$, $\|\lambda\|_2 < 1$ we have

$$\langle\varphi, z_\lambda\rangle = \langle e_1 \otimes e_2, z_\lambda\rangle - \langle e_2 \otimes e_1, z_\lambda\rangle = \lambda_1\lambda_2 - \lambda_2\lambda_1 = 0.$$

More generally if $f \in F(k, \Lambda)$ and $\pi \in \Pi_k$, (π is a permutation on $\{1, 2, \cdots, k\}$), let $\pi(f) \in F(k, \Lambda)$ be defined by $\pi(f)(i) = f(\pi(i))$. Then $e_f - e_{\pi(f)} \in \mathcal{N}$.

PROPOSITION 3.6. *$F^\infty/\mathcal{N}^\infty$ is a commutative algebra.*

Proof. Let $\varphi = \sum_{f\in\mathcal{F}} a_f e_f$, $\psi = \sum_{g\in\mathcal{F}} b_g e_g$ be elements in F^∞. We want to prove that $\varphi \otimes \psi - \psi \otimes \varphi \in \mathcal{N}$. Notice that

$$\varphi \otimes \psi - \psi \otimes \varphi = \sum_{f\in\mathcal{F}} \sum_{g\in\mathcal{F}} a_f b_g [e_f \otimes e_g - e_g \otimes e_f].$$

Let $\lambda \in \mathbf{C}^n$, $\|\lambda\|_2 < 1$. Since $\langle e_f \otimes e_g - e_g \otimes e_f, z_\lambda\rangle = 0$ we get that $\langle\varphi \otimes \psi - \psi \otimes \varphi, z_\lambda\rangle = 0$. Since λ is arbitrary we finish. $\blacksquare$

4. REFLEXIVITY RESULTS. Let $\mathcal{H}$ be a Hilbert space and $B(\mathcal{H})$ be the algebra of all bounded operators on $\mathcal{H}$. If $A \in B(\mathcal{H})$ then the set of all invariant subspaces of A is denoted by Lat A. For any $\mathcal{U} \subset B(\mathcal{H})$ we define

$$\text{Lat } \mathcal{U} = \bigcap_{A \in \mathcal{U}} \text{Lat } A.$$

If $\mathcal{S}$ is any collection of subspaces of $\mathcal{H}$, then Alg $\mathcal{S} := \{A \in B(\mathcal{H}) : \mathcal{S} \subset \text{Lat } A\}$. The algebra $\mathcal{U} \subset B(\mathcal{H})$ is reflexive if $\mathcal{U} = \text{Alg Lat } \mathcal{U}$.

The main theorems of this section are the following.

THEOREM 4.1. *The algebra* $\{\varphi(S_1, S_2, \cdots, S_n) : \varphi \in F^\infty\}$ *is reflexive.*

THEOREM 4.2. *If* $\mathcal{U}$ *is a strongly closed subalgebra of* $\{\varphi(S_1, \cdots, S_n);\ \varphi \in F^\infty\}$ *containing the identity then* $\mathcal{U}$ *is reflexive.*

Proof of Theorem 4.1 Let $A \in \text{Alg Lat}\, F^\infty$. For every φ inner we have that $\mathcal{F}^2(H_n) \otimes \tilde{\varphi} \in \text{Lat}\, F^\infty$. Then $A[\mathcal{F}^2(H_n) \otimes \tilde{\varphi}] \subset \mathcal{F}^2(H_n) \otimes \tilde{\varphi}$. In particular we have

$$A\tilde{\varphi} = \psi_\varphi \otimes \tilde{\varphi} \qquad \text{for some} \quad \psi_\varphi \in \mathcal{F}^2(H_n). \tag{18}$$

For every $f \in \mathcal{F}$ we can find ψ_f such that

$$Ae_f = \psi_f \otimes e_f.$$

(Notice that both e_f and $\tilde{e}_f$ are inner and we do not have to carry the tilde).

Let $k \geq 1$. Example 4 of Section 3 tells that $x_k = a_k \sum_{f \in F(k,\Lambda)} e_f$, where $a_k = [\text{card}(F(k,\Lambda))]^{-1/2}$, is inner (notice that $x_k = \tilde{x}_k$). Using (18) we have that

$$Ax_k = \psi_k \otimes x_k = a_k \sum_{f \in F(k,\Lambda)} \psi_k \otimes e_f. \tag{19}$$

On the other hand, we have that

$$Ax_k = A\Bigg(a_k \sum_{f \in F(k,\Lambda)} e_f\Bigg) = a_k \sum_{f \in F(k,\Lambda)} Ae_f = a_k \sum_{f \in F(k,\Lambda)} \psi_f \otimes e_f. \tag{20}$$

We claim that for every $f \in F(k,\Lambda)$, $\psi_k = \psi_f$. Fix $f \in F(k,\Lambda)$ and let P_f be the orthogonal projection onto $\mathcal{F}^2(\mathcal{H}_n) \otimes e_f$. From (19) we get that $P_f(Ax_k) = a_k \psi_k \otimes e_f$ (the other terms are zero), and from (20) we get that $P_f(Ax_k) = a_k \psi_f \otimes e_f$. Therefore, $\psi_k \otimes e_f = \psi_f \otimes e_k$ and then $\psi_k = \psi_f$.

Let $k > 1$. Example 5 of Section 3 tells us that $y = \frac{1}{\sqrt{2}}[e_1 + e_2^k]$ is inner (notice that $y = \tilde{y}$). Using (18) we have that

$$Ay = \psi_y \otimes y = \frac{1}{\sqrt{2}}[\psi_y \otimes e_1 + \psi_y \otimes e_2^k]. \tag{21}$$

On the other hand,

$$Ay = \frac{1}{\sqrt{2}}[Ae_1 + Ae_2^k] = \frac{1}{\sqrt{2}}[\psi_1 \otimes e_1 + \psi_k \otimes e_2^k]. \tag{22}$$

The last equality is clear since e_1 has *one* letter and e_2^k has k letters. Combining (21) and (22) we conclude that $\psi_k = \psi_1$.

Summarizing we have that $Ae_0 = \psi_0$ for some $\psi_0 \in F^2(H_n)$, and if $f \in F(k,\Lambda)$, $k \geq 1$, then $Ae_f = \psi_1 \otimes e_f$. It is easy to see that $\psi_1 \in F^\infty$.

Let $B = A - \psi_1(S_1, S_2, \cdots, S_n)$. We still have that $B \in \text{Alg Lat } F^\infty$ and if we set $\psi = \psi_0 - \psi_1$,

$$Be_0 = \psi$$
$$Be_f = 0 \qquad \text{if} \quad f \neq 0.$$

We want to prove that $B = 0$. This gives us that $A = \psi_1 \in F^\infty$.

Equation (18) applies to B as well. That is, if φ is inner, then $B\tilde{\varphi} = \psi_\varphi \otimes \tilde{\varphi}$. On the other hand it is clear that $B\tilde{\varphi} = \langle e_0, \tilde{\varphi}\rangle\psi$. Hence, if $\langle e_0, \tilde{\varphi}\rangle \neq 0$, we have that

$$\psi = \frac{1}{\langle e_0, \tilde{\varphi}\rangle}\psi_\varphi \otimes \tilde{\varphi} \in F^2(H_n) \otimes \tilde{\varphi}, \quad \text{and} \quad \tilde{\psi} \in \varphi \otimes \mathcal{F}^2(\mathcal{H}_n).$$

It is easy to see that $\varphi(f,\mu)$ (the Möbius maps of Example 7, Section 3) satisfy $\langle e_0, \tilde{\varphi}(f,\mu)\rangle \neq 0$ for any $f \in \mathcal{F}$ and $0 < |\mu| < 1$. This implies that

$$\tilde{\psi} \in \bigcap \{\varphi(f,\mu) \otimes \mathcal{F}^2(\mathcal{H}_n) : f \in \mathcal{F}, 0 < |\mu| < 1\}.$$

By Lemma 3.2 we conclude that $\psi = 0$. ■

REMARK. The previous proof does not work in the commutative case. We are using the fact that e_1 and e_2 are non-commutative to conclude that $\frac{1}{\sqrt{2}}[e_1 + e_2^k]$ is inner. However, $\frac{1}{\sqrt{2}}[z + z^k]$ is not inner in H^∞.

The following notation will be useful in the proof of Theorem 4.2: If $A \in B(\mathcal{H})$ and m is a positive integer, then $\mathcal{H}^{(m)}$ denotes the direct sum of m copies of $\mathcal{H}$ and $A^{(m)}$ stands for the direct sum of m copies of A. If $\mathcal{U} \subset B(\mathcal{H})$, then $\mathcal{U}^{(m)} := \{A^{(m)} : A \in \mathcal{U}\}$.

According to [RR, Theorem 7.1], if $\mathcal{U}$ is an algebra of operators containing the identity, then the closure of $\mathcal{U}$ in the strong operator topology is

$$\{B : \text{Lat } \mathcal{U}^{(m)} \subset \text{Lat } B^{(m)} \quad \text{for } m = 1, 2, 3, \cdots\}.$$

To prove Theorem 4.2 we need the following

THEOREM 4.3. *If* $A, B \in \{\varphi(S_1, \cdots, S_n); \quad \varphi \in F^\infty\}$ *are such that Lat* $A \subset$ *Lat* B, *then*

$$\text{Lat } A^{(m)} \subset \text{Lat } B^{(m)}, \quad \text{for any } m = 1, 2, \cdots.$$

Proof. It will enough to show that every cyclic invariant subspace of $A^{(m)}$ is invariant under $B^{(m)}$. Let $x \neq 0$ be an element in $F^2(H_n)^{(m)}$ and let $\mathcal{M} = \bigvee_{f \in \mathcal{F}} S_f^{(m)} x$. According to Lemma 2.5 there is a unitary operator $U : F^2(H_n) \to \mathcal{M}$ such that

$$S_i^{(m)}|_{\mathcal{M}} = U S_i U^{-1}, \quad \text{for any } i = 1, 2, \cdots.$$

Since A and B are strong limits of polynomials in $S_1, \cdots, S_n$ it follows that

$$\begin{aligned} A^{(m)}|_{\mathcal{M}} &= U A U^{-1}, \quad \text{and} \\ B^{(m)}|_{\mathcal{M}} &= U B U^{-1}. \end{aligned} \tag{23}$$

Let $\mathcal{G}$ be the smallest invariant subspace of $A^{(m)}$ containing x, i.e.,

$$\mathcal{G} = \bigvee_{f \in \mathcal{F}} A_f^{(m)} x.$$

Since $A^{(m)} = \varphi(S_1^{(m)}, \cdots, S_n^{(m)})$ for some $\varphi \in F^\infty$ it follows that $\mathcal{G} \subset \mathcal{M}$. The relation (23) and the fact that $A^{(m)}\mathcal{G} \subset \mathcal{G}$ implies $UAU^{-1}(\mathcal{G}) \subset \mathcal{G}$ whence $A(U^{-1}\mathcal{G}) \subset U^{-1}\mathcal{G}$. But Lat $A \subset$ Lat B implies $B(U^{-1}\mathcal{G}) \subset U^{-1}\mathcal{G}$. Therefore, $UBU^{-1}(\mathcal{G}) \subset \mathcal{G}$ which together with (23) show that $B^{(m)}\mathcal{G} \subset \mathcal{G}$. ■

COROLLARY 4.4. *Let* $A, B \in \{\varphi(S_1, \cdots, S_n) : \quad \varphi \in F^\infty\}$ *be such that Lat* $A \subset$ *Lat* B, *then* B *belongs to the strongly closed algebra generated by* A *and the identity.*

The proof of the following theorem is analogue to the proof of the Theorem 4.3 if we replace A with any subset $\mathcal{A} \subset \{\varphi(S_1, \cdots, S_n) : \quad \varphi \in F^\infty\}$. We omit the proof.

THEOREM 4.5. *If* $B \in \{\varphi(S_1, \cdots, S_n) : \quad \varphi \in F^\infty\}$ *is such that Lat* $\mathcal{A} \subset$ *Lat* B *then*

$$\text{Lat } \mathcal{A}^{(m)} \subset \text{ Lat } B^{(m)}, \quad \text{for any } m = 1, 2, \cdots.$$

Moreover, B *belongs to the strongly closed algebra generated by* $\mathcal{A}$ *and the identity.*

Proof of Theorem 4.1. Let $B \in B(\mathcal{F}^2(\mathcal{H}_n))$ be such that

$$\text{Lat } \mathcal{U} \subset \text{Lat } B.$$

It is clear that

$$\text{Lat } \{S_1, \cdots, S_n\} \subset \text{ Lat } \{\varphi(S_1, \cdots, S_n); \quad \varphi \in F^\infty\} \subset \text{ Lat } \mathcal{U} \subset \text{ Lat } B.$$

According to Theorem 4.1, we have

$$B \in \{\varphi(S_1, \cdots, S_n); \quad \varphi \in F^\infty\}.$$

Now Theorem 4.5 implies that $B \in \mathcal{U}$. ■

Let us recall from [Po2] that an operator $T \in B(\mathcal{F}^2(\mathcal{H}_n))$ is called multi-analytic if $TS_i = S_iT$ for each $i \in \Lambda = \{1, 2, \cdots, n\}$. The following result is an easy consequence of Theorem 4.2 and the characterization of the multi-analytic operators in terms of their symbols [Po4].

COROLLARY 4.6. *Any strongly closed subalgebra of multi-analytic operators containing the identity is reflexive.*

Let us remark that in the particular case when $\Lambda = \{1\}$ we find again the result of Sarason [S].

5. OPEN QUESTIONS. We finish this paper with some questions.

Let Inv (F^∞) be the group of invertible elements in F^∞, and $\mathcal{G}_0$ be the connected component in Inv (F^∞) which contains the identity. From the general theory of Banach algebras [D] we know that the collection of finite products of elements in $\exp \mathcal{F}^\infty$ is $\mathcal{G}_0$ and that Inv $(F^\infty)/\mathcal{G}_0$ is a discrete group.

PROBLEM 1. *Characterize Inv $(F^\infty)/\mathcal{G}_0$.*

PROBLEM 2. *Characterize the proper maximal invariant subspaces for the Λ-orthogonal shift of $\mathcal{F}^2(\mathcal{H}_n)$.*

It is clear that all of the $\mathcal{M}_\lambda$ of Example 8, Section 3 are maximal. However, there are many invariant subspaces for the shift that are not inside any of them.

Take, for instance, $e_f = e_1 \otimes e_2 \otimes \cdots \otimes e_n$ and $\mu \in \mathbf{C}$, $\frac{1}{\sqrt{n}} < |\mu| < 1$. We claim that $\mathcal{M} = \mathcal{F}^2(\mathcal{H}_n) \otimes \tilde{\varphi}(f, \mu)$ is not a subset of $\mathcal{M}_\lambda$ for any $\lambda \in \mathbf{C}^n$, $\|\lambda\|_2 < 1$. (See Example 6, Section 3 for the notation).

It is clear that $\mathcal{M} \subset \mathcal{M}_\lambda$ if and only if $z_\lambda \in \mathcal{M}^\perp$ and since $z_\lambda = \tilde{z}_\lambda$ it is equivalent to $z_\lambda \in [\varphi(f, \mu) \otimes \mathcal{F}^2(\mathcal{H}_n)]^\perp$. Using (16) we get that $\mathcal{M} \subset \mathcal{M}_\lambda$ if and only if

$$\forall g \in \mathcal{F}, \quad \langle e_f \otimes e_g \rangle = \mu \langle e_g, z_\lambda \rangle, \qquad \text{equivalently}$$
$$\forall g \in \mathcal{F}, \quad \lambda_f \lambda_g = \mu \lambda_g.$$

Since $\lambda_0 = 1$, we have that $\lambda_f = \lambda_1 \lambda_2 \cdots \lambda_n = \mu$. However, $\sum_{i \le n} |\lambda_i|^n < 1$ implies that

$$|\mu| = |\lambda_1 \lambda_2 \cdots \lambda_n| < \frac{1}{\sqrt{n}},$$

contradicting the condition on μ.

PROBLEM 3. *Characterize the proper maximal subspaces of $\mathcal{F}^2(\mathcal{H}_n)$ of the form $\varphi \otimes \mathcal{F}^2(\mathcal{H}_n)$, φ inner.*

It follows from Corollary 2.7 that if φ_1, φ_2 are inner and $\varphi_1 \otimes \mathcal{F}^2(\mathcal{H}_n) \subset \varphi_2 \otimes \mathcal{F}^2(\mathcal{H}_n)$ then there exists φ_3 inner such that $\varphi_1 = \varphi_2 \otimes \varphi_3$. Then the maximal subspaces of the form $\varphi \otimes \mathcal{F}^2(\mathcal{H}_n)$ correspond to "prime" inner functions.

PROBLEM 4. *Is every inner function φ a product of "prime" ones? The question refers mainly to convergence.*

REFERENCES

[A] A. Arias, *Completely bounded isomorphisms of operator algebras*, Proceedings of the AMS (to appear)

[D] R. G. Douglas, *Banach algebra techniques in operator theory*, Academic Press, New York, (1972).

[Po1] G. Popescu, *Characteristic functions for infinite sequences of noncommuting operators*, J. Operator Theory, **22** (1989), 51–71.

[Po2] G. Popescu, *Multi-analytic operators and some factorization theorems* Indiana Univ. Math. J. **38** (1989), 693–710.

[Po3] G. Popescu, *Von Neumann inequality for $(B(H)^n)_1$*, Math. Scan. **68** (1991), 292–304.

[Po4] G. Popescu, *Multi–analytic operators on Fock spaces* (preprint).

[RR] H. Radjavi, P. Rosenthal, *Invariant subspaces*, Springer-Verlag, Berlin (1973)

[S] D. Sarason, *Invariant subspaces and unstarred operator algebras*, Pacific J. Math. **17** (1966), 511–517.

[vN] J. von Neumann, *Eine Spectraltheorie für allgemeine Operatoren eines unitären Raumes*, Math. Nachr. **4** (1951), 258–281.

Alvaro Arias[1]
Division of Mathematics, Computer Science and Statistics,
The University of Texas at San Antonio,
San Antonio, TX 78249, U.S.A.
e-mail arias@ringer.cs.utsa.edu

Gelu Popescu
Division of Mathematics, Computer Science and Statistics,
The University of Texas at San Antonio,
San Antonio, TX 78249, U.S.A.
email gpopescu@ringer.cs.utsa.edu

[1] The first author was supported in part by NSF DMS 93-21369
1991 *Mathematics Subject Classification.* Primary 47D25, Secondary 32A35, 47A67.