

# OPERATOR HILBERT SPACES WITHOUT THE OPERATOR APPROXIMATION PROPERTY

ALVARO ARIAS

ABSTRACT. We use a technique of Szankowski [S] to construct operator Hilbert spaces that do not have the operator approximation property, including an example in a noncommutative  $L_p$  space for  $p \neq 2$ .

## 1. INTRODUCTION AND PRELIMINARIES

A Banach space  $X$  has the approximation property, or AP, if the identity operator on  $X$  can be approximated uniformly on compact subsets of  $X$  by linear operators of finite rank. In the 50's, Grothendieck [G] investigated this property and found several equivalent statements. For example, he proved that  $X$  has the AP iff the natural map  $J : X^* \hat{\otimes} X \rightarrow X^* \check{\otimes} X$  is one-to-one ( $\hat{\otimes}$  is the projective tensor product of Banach spaces and  $\check{\otimes}$  is the injective tensor product of Banach spaces). However, it remained unknown if every Banach space had the AP until Enflo [E] constructed the first counter example in the early 70's. In [S], Szankowski gave a very explicit example of a subspace of  $\ell_p$ ,  $1 < p < 2$ , without the AP. He considered  $X = (\sum_{n=1}^{\infty} \oplus \ell_2^n)_p$ , which is isomorphic to  $\ell_p$ , and defined  $Z$  to be the closed span of some vectors of length six. He then used a clever combinatorial argument to exploit the difference between the 2-norm of the blocks and the  $p$ -norm of the sum to prove that  $Z$  fails the approximation property. Szankowski's technique is fairly general. In the second section of this paper we will use it to show that the  $\ell_2$ -sum (as defined in [P2]) of row operator spaces has a subspace without the operator space version of the approximation property, or OAP. Since this subspace is a Hilbert space at the Banach space level, it has the Banach approximation property and even a basis. Thus, this is an example of an operator space with the AP but without the OAP. This answers a question of J. Kraus. Furthermore, this is also the first example of an operator space with a basis but without a complete basis. M. Junge suggested that a similar construction using rows in the Schatten  $p$ -class  $S_p$  and Rademacher functions in  $L_p[0, 1]$  could lead to an example of a Hilbertian subspace of  $L_p[S_p]$  failing the OAP. In the third section we verify that this is indeed possible. These are new examples of Hilbertian subspaces of noncommutative  $L_p$  spaces that are not completely complemented, even if  $p > 2$ .

An operator space  $E$  is a Banach space  $E$  with an isometric embedding into  $B(H)$ , the set of all bounded operators on a Hilbert space  $H$ . Or, equivalently, an operator space  $E$  is a closed subspace of  $B(H)$ . If  $E \subset B(H_1)$  and  $F \subset B(H_2)$  are operator spaces, their *minimal tensor product*  $E \otimes_{\min} F$  is the closure of the algebraic

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tensor product  $E \otimes F$  in  $B(H_1 \otimes_2 H_2)$ . A linear map  $u : E \rightarrow F$  is *completely bounded*, or cb, if for every operator space  $G$ , the map  $1_G \otimes u : G \otimes_{\min} E \rightarrow G \otimes_{\min} F$  is bounded. The completely bounded norm of  $u$ , or  $\|u\|_{cb}$ , is the supremum of  $\|1_G \otimes u\|$ , where  $G$  runs over all operator spaces  $G$ . It turns out that it is enough to verify that  $1_G \otimes u$  is bounded when  $G$  is  $K(\ell_2)$ , the set of all compact operators on the Hilbert space  $\ell_2$ , and that  $\|u\|_{cb} = \|1_{K(\ell_2)} \otimes u\|$ . The set of all cb-maps from  $E$  to  $F$  is denoted by  $CB(E, F)$ . Independently of each other, Blecher and Paulsen [BP] and Effros and Ruan [ER1] gave  $E^*$ , the Banach space dual of  $E$ , an operator space structure that gives  $E^* \otimes_{\min} F$  the norm induced by  $CB(E, F)$ . This indicates that the minimal tensor product is the operator space analogue of the injective tensor product of Banach spaces. In the same papers, Blecher and Paulsen [BP] and Effros and Ruan [ER1] introduced the operator space analogue of the projective tensor product. This is denoted by  $E \hat{\otimes} F$  and satisfies  $(E \hat{\otimes} F)^* = CB(E, F^*)$ . We refer to [ER3], [J], and [P3] for more information about operator spaces.

It is well known that the compact subsets of a Banach space  $X$  are contained in the convex hull of null sequences in  $X$ . Since there is a correspondence between null sequences in  $X$  and elements of  $c_0 \hat{\otimes} X$ , it is easy to see that  $X$  has the AP iff for every  $u \in c_0 \hat{\otimes} X$  and every  $\epsilon > 0$ , there exists a finite rank operator on  $X$  such that  $\|u - (I \otimes T)(u)\| < \epsilon$ . Based on this observation, Effros and Ruan [EF2] said that an operator space  $V$  has the operator approximation property, or OAP, if for every  $u \in K(H) \otimes_{\min} V$  and every  $\epsilon > 0$ , there exists a finite rank operator  $T$  on  $V$  such that  $\|u - (I \otimes T)(u)\| < \epsilon$ . They proved that an operator space  $V$  has the OAP if and only if the natural map  $J : V^* \hat{\otimes} V \rightarrow V^* \otimes_{\min} V$  is one-to-one.

The following criterion allows us to check that  $J$  is not one-to-one, when  $V$  fails the OAP.

**Enflo's Criterion.** *If there exist a sequence of finite rank operators  $\beta_n \in V^* \otimes V$  satisfying:*

- (i)  $\text{trace}(\beta_n) = 1$  for every  $n \in \mathbb{N}$ ,
  - (ii)  $\|\beta_n\|_{V^* \otimes_{\min} V} \rightarrow 0$  as  $n \rightarrow \infty$ , and
  - (iii)  $\sum_{n=1}^{\infty} \|\beta_n - \beta_{n-1}\|_{V^* \hat{\otimes} V} < \infty$ ,
- then  $V$  does not have the OAP.

Indeed,  $\beta = \beta_1 + \sum_{n=2}^{\infty} \beta_n - \beta_{n-1} = \lim_n \beta_n$  belongs to  $V^* \hat{\otimes} V$  by (iii).  $J\beta = 0$  by (ii). And since  $\text{tr}(\beta) = 1$ ,  $\beta$  is not zero. Hence  $J$  is not one-to-one and  $V$  fails the OAP.

## 2. AN EXAMPLE USING COMPLEX INTERPOLATION

For each  $n \in \mathbb{N}$ , let  $\Delta_n$  be a partition of  $\sigma_n = \{2^n, 2^n + 1, 2^n + 2, \dots, 2^{n+1} - 1\}$ . Then  $\{B \in \Delta_n : n \in \mathbb{N}\}$  is a partition of  $\mathbb{N}$ . For each  $B \in \Delta_n$ , let  $R_B$  be the row Hilbert space with orthonormal basis  $\{e_j : j \in B\}$ . We define  $X$  to be the  $\ell_2$ -sum of these row spaces. More precisely,  $X$  is the complex interpolation space between  $(\sum_{n=1}^{\infty} \sum_{B \in \Delta_n} \oplus R_B)_{\infty}$  and  $(\sum_{n=1}^{\infty} \sum_{B \in \Delta_n} \oplus R_B)_1$  of parameter  $\theta = \frac{1}{2}$  (see [P2], page 34). That is,

$$X = \left( \sum_{n=1}^{\infty} \sum_{B \in \Delta_n} \oplus R_B \right)_{\ell_2} = \left( \left( \sum_{n=1}^{\infty} \sum_{B \in \Delta_n} \oplus R_B \right)_{\infty}, \left( \sum_{n=1}^{\infty} \sum_{B \in \Delta_n} \oplus R_B \right)_1 \right)_{\frac{1}{2}}.$$

At the Banach space level,  $X$  is a Hilbert space with orthonormal basis  $\{e_i : i \in \mathbb{N}\}$ . But at the operator space level,  $X$  is a combination of row Hilbert spaces and  $OH$ ,

the self dual operator Hilbert space introduced by Pisier in [P1]. If  $A \subset \mathbb{N}$ , let  $X_A = \overline{\text{span}}\{e_i : i \in A\} \subset X$ . It follows from the definition of  $X$  that if there exists  $n \in \mathbb{N}$  such that  $A \subset B$  for some  $B \in \Delta_n$ , then  $X_A$  is completely isometric to  $R_A$ , the row Hilbert space with orthonormal basis  $\{e_i : i \in A\}$ . And if for each  $n \in \mathbb{N}$ ,  $A$  has at most one point from each element in  $\Delta_n$ , (i.e,  $\text{card}(A \cap B) \leq 1$  for every  $B \in \Delta_n, n \in \mathbb{N}$ ), then  $X_A$  is completely isometric to  $OH_A$ .

Let  $Z$  be the closed subspace of  $X$  spanned by

$$z_i = e_{2i} - e_{2i+1} + e_{4i} + e_{4i+1} + e_{4i+2} + e_{4i+3}, \quad i = 1, 2, \dots$$

**Theorem 1.** *With the appropriate selection of  $\Delta_n$ ,  $Z$  does not have the OAP.*

For each  $i \in \mathbb{N}$ , let  $z_i^* = \frac{1}{2}(e_{2i}^* - e_{2i+1}^*)$ , where the  $e_i^*$ 's are biorthogonal to the  $e_i$ 's. Then let

$$\beta_n = \frac{1}{2^n} \sum_{i \in \sigma_n} z_i^* \otimes z_i \quad \text{for } n \geq 2.$$

We need to check that the  $\beta_n$ 's satisfy the conditions of Enflo's criterion.

Condition (i)

This is trivially verified. Since  $z_i^*(z_i) = 1$  for every  $i \geq 1$ , we see that  $\text{trace}(\beta_n) = (1/2^n) \sum_{i \in \sigma_n} z_i^*(z_i) = (1/2^n)|\sigma_n| = 1$ .

Condition (ii)

Since  $\|\beta_n\|_{Z^* \otimes_{\text{min}} Z} \leq \|\beta_n\|_{X^* \otimes_{\text{min}} X} = \|\beta_n\|_{cb}$ , we will estimate the cb-norm of  $\beta_n : X \rightarrow X$ . However, it follows from the definition of  $\beta_n$  that we only need to estimate the cb-norm of  $\beta_n : X_{\sigma_{n+1}} \rightarrow X_{\sigma_{n+1} \cup \sigma_{n+2}}$ , where  $X_{\sigma_k} = \text{span}\{e_i : i \in \sigma_k\}$ . Let  $I_1 : X_{\sigma_{n+1}} \rightarrow R_{\sigma_{n+1}}$  and  $I_2 : R_{\sigma_{n+1} \cup \sigma_{n+2}} \rightarrow X_{\sigma_{n+1} \cup \sigma_{n+2}}$  be the formal identity maps, and let  $\tilde{\beta}_n : R_{\sigma_{n+1}} \rightarrow R_{\sigma_{n+1} \cup \sigma_{n+2}}$  be  $\tilde{\beta}_n = \frac{1}{2^n} \sum_{i \in \sigma_n} z_i^* \otimes z_i$  (that is,  $\tilde{\beta}_n$  has the same matrix representation of  $\beta_n$ , but it is defined on row operator spaces). Then  $\beta_n = I_2 \circ \tilde{\beta}_n \circ I_1$ . Since the  $z_i$ 's,  $i \in \sigma_n$ , have disjoint support, the  $z_i^*$ 's,  $i \in \sigma_n$ , have also disjoint support, and the row spaces are homogeneous, it is easy to see that  $\|\tilde{\beta}_n\|_{cb} = \|\tilde{\beta}_n\| = \frac{1}{2^{n+1}}\sqrt{3}$ . We will prove condition (ii) by checking that  $\|I_1\|_{cb}\|I_2\|_{cb} \leq \sqrt{2^{n+2}}$ .

From the definition of  $X$ , we see that  $X_{\sigma_{n+1}}$  is equal to  $(\sum_{B \in \Delta_{n+1}} \oplus R_B)_{\ell_2}$ , the complex interpolation space  $((\sum_{B \in \Delta_{n+1}} \oplus R_B)_{\infty}, (\sum_{B \in \Delta_{n+1}} \oplus R_B)_1)_{\frac{1}{2}}$ . It is easy to check that  $\|I_1 : (\sum_{B \in \Delta_{n+1}} \oplus R_B)_{\infty} \rightarrow R_{\sigma_{n+1}}\|_{cb} \leq \sqrt{|\Delta_{n+1}|}$  and that  $\|I_1 : (\sum_{B \in \Delta_{n+1}} \oplus R_B)_1 \rightarrow R_{\sigma_{n+1}}\|_{cb} \leq 1$ . Therefore  $\|I_1\|_{cb} \leq (|\Delta_{n+1}|)^{\frac{1}{4}}$ . Similarly,  $\|I_2\|_{cb} \leq (|\Delta_{n+1}| + |\Delta_{n+2}|)^{\frac{1}{4}}$ . Since  $|\Delta_k| \leq 2^k$  for every  $k \in \mathbb{N}$ , we see that  $\|I_1\|_{cb}\|I_2\|_{cb} \leq \sqrt{2^{n+2}}$ .

Condition (iii)

Using the fact that  $z_i^* = \frac{1}{4}(e_{4i}^* + e_{4i+1}^* + e_{4i+2}^* + e_{4i+3}^*)$  on  $Z$ , we get that

$$\begin{aligned} \beta_n - \beta_{n-1} &= \frac{1}{2^{n+1}} \sum_{i \in \sigma_n} (e_{2i}^* - e_{2i+1}^*) \otimes (e_{2i} - e_{2i+1} + e_{4i} + e_{4i+1} + e_{4i+2} + e_{4i+3}) \\ &\quad - \frac{1}{2^{n+1}} \sum_{i \in \sigma_{n-1}} (e_{4i}^* + e_{4i+1}^* + e_{4i+2}^* + e_{4i+3}^*) \otimes (e_{2i} - e_{2i+1} + e_{4i} + e_{4i+1} + e_{4i+2} + e_{4i+3}) \\ &= \frac{1}{2^{n+1}} \sum_{i \in \sigma_{n-1}} \left\{ \begin{array}{l} e_{4i}^* \otimes (e_{4i} - e_{4i+1} + e_{8i} + e_{8i+1} + e_{8i+2} + e_{8i+3} - e_{2i} + e_{2i+1} - e_{4i} - e_{4i+1} - e_{4i+2} - e_{4i+3}) \\ e_{4i+1}^* \otimes (-e_{4i} + e_{4i+1} - e_{8i} - e_{8i+1} - e_{8i+2} - e_{8i+3} - e_{2i} + e_{2i+1} - e_{4i} - e_{4i+1} - e_{4i+2} - e_{4i+3}) \\ e_{4i+2}^* \otimes (e_{4i+2} - e_{4i+3} + e_{8i+4} + e_{8i+5} + e_{8i+6} + e_{8i+7} - e_{2i} + e_{2i+1} - e_{4i} - e_{4i+1} - e_{4i+2} - e_{4i+3}) \\ e_{4i+3}^* \otimes (-e_{4i+2} + e_{4i+3} - e_{8i+4} - e_{8i+5} - e_{8i+6} - e_{8i+7} - e_{2i} + e_{2i+1} - e_{4i} - e_{4i+1} - e_{4i+2} - e_{4i+3}) \end{array} \right\} \end{aligned}$$

Note that after cancellation, each of the vectors in the parenthesis has nine terms. Two of them cancel out and two are equal. Then we can write each of them as a linear combination of nine vector basis. Eight of them have coefficients equal to  $\pm 1$  and the other has a coefficient equal to  $\pm 2$ .

Szankowski defined nine functions  $f_k : \mathbb{N} \rightarrow \mathbb{N}, k \leq 9$ , to index these vectors. Let  $n = 4i + l$  and  $l = 0, 1, 2, 3$ . Then  $f_1(4i + l) = 2i$  and  $f_2(4i + l) = 2i + 1$ . For  $k = 3, 4, 5$ ,  $f_k(4i + l) = 4i + [(l + 1) \bmod 4]$ . For  $l = 0, 1$ ,  $f_6(4i + l) = 8i$ ,  $f_7(4i + l) = 8i + 1$ ,  $f_8(4i + l) = 8i + 2$ , and  $f_9(4k + l) = 8i + 3$ . And finally, for  $l = 2, 3$ ,  $f_6(4i + l) = 8i + 4$ ,  $f_7(4i + l) = 8i + 5$ ,  $f_8(4i + l) = 8i + 6$ , and  $f_9(4k + l) = 8i + 7$ . Then we have

$$\beta_n - \beta_{n-1} = \frac{1}{2^{n+1}} \sum_{j \in \sigma_{n+1}} e_j^* \otimes y_j,$$

where  $y_j = \sum_{k=1}^9 \lambda_{j,k} e_{f_k(j)} \in Z$ . Recall that eight of the  $\lambda_{j,k}$ 's have absolute value equal to one, and one has absolute value equal to 2.

The following Lemma of Szankowski provides the key combinatorial argument (see [S] and [LT, page 108]).

**Lemma 2 (Szankowski).** *There exist partitions  $\Delta_n$  and  $\nabla_n$  of  $\sigma_n$  into disjoint sets, and a sequence  $m_n \geq 2^{\frac{n}{8}-2}$ ,  $n = 2, 3, \dots$ , so that*

- (1)  $\forall A \in \nabla_n, m_n \leq \text{card}(A) \leq 2m_n$ ,
- (2)  $\forall A \in \nabla_n, \forall B \in \Delta_n, \text{card}(A \cap B) \leq 1$ ,
- (3)  $\forall A \in \nabla_n, \forall 1 \leq k \leq 9, f_k(A)$  is contained in an element of  $\Delta_{n-1}, \Delta_n$ , or  $\Delta_{n+1}$ .

(Notice that  $f_k(\sigma_n) \subset \sigma_{n-1}$  for  $k = 1, 2$ ,  $f_k(\sigma_n) \subset \sigma_n$  for  $k = 3, 4, 5$ , and  $f_k(\sigma_n) \subset \sigma_{n+1}$  for  $k = 6, 7, 8, 9$ ).

Since  $\nabla_{n+1}$  is a partition of  $\sigma_{n+1}$ , we have that

$$(1) \quad \beta_n - \beta_{n-1} = \frac{1}{2^{n+1}} \sum_{A \in \nabla_{n+1}} \left[ \sum_{j \in A} e_j^* \otimes y_j \right].$$

**Lemma 3.** *For every  $A \in \nabla_{n+1}$ ,  $\|\sum_{j \in A} e_j^* \otimes y_j\|_{Z^* \hat{\otimes} Z} \leq 18(\text{card}(A))^{\frac{3}{4}}$ .*

The last condition of Enflo's criterion follows immediately from (1), Lemma 2, and Lemma 3. Indeed,

$$\begin{aligned} \|\beta_n - \beta_{n-1}\|_{Z^* \hat{\otimes} Z} &\leq \frac{1}{2^{n+1}} \text{card}(\nabla_{n+1}) 18 \max_{A \in \nabla_{n+1}} \text{card}(A)^{\frac{3}{4}} \\ &\leq \frac{1}{2^{n+1}} \frac{2^{n+1}}{m_{n+1}} 18 (2m_{n+1})^{\frac{3}{4}} \leq \frac{36}{m_{n+1}^{\frac{1}{4}}}, \end{aligned}$$

which is clearly summable.

We only need to prove Lemma 3. For this, we need the result of Pisier (see remark 2.11 of [P1]) that  $CB(R_n, OH_n) = S_4^n$ , where  $S_4^n$  is the Schatten 4-class. Consequently, if  $S : OH_n \rightarrow R_n$ , then  $\|S\|_{OH_n \hat{\otimes} R_n} = \|S\|_{S_{4/3}^n}$ . In particular, if  $I : OH_n \rightarrow R_n$  is the formal identity,  $\|I\|_{OH_n \hat{\otimes} R_n} = n^{3/4}$ .

**Proof of Lemma 3.** The element  $\gamma = \sum_{j \in A} e_j^* \otimes y_j \in X^* \hat{\otimes} Z$  induces a finite rank map  $\gamma : X \rightarrow Z$ . The restriction of  $\gamma$  to  $Z$  is the map  $\alpha = \gamma|_Z : Z \rightarrow Z$ , which clearly satisfies  $\alpha = \sum_{j \in A} q(e_j^*) \otimes y_j \in Z^* \hat{\otimes} Z$ , where  $q = (\iota_Z)^* : X^* \rightarrow Z^*$  is the adjoint of the inclusion  $\iota_Z : Z \rightarrow X$ . Since  $(Z^* \hat{\otimes} Z)^* = CB(Z^*, Z^*)$  we have that  $\|\alpha\|_{Z^* \hat{\otimes} Z} = \sup\{|\langle T, \alpha \rangle| : T : Z^* \rightarrow Z^*, \|T\|_{cb} \leq 1\}$ , where  $\langle \cdot, \cdot \rangle$  is the trace duality.

We will see that we can factor  $\alpha$  through the formal identity map  $I : OH_A \rightarrow R_A$ , where  $R_A$  is the row Hilbert space with basis  $\{\delta_j : j \in A\}$ . Recall that the projective tensor norm of  $I : OH_A \rightarrow R_A$ , viewed as an element of  $OH_A \hat{\otimes} R_A$ , is equal to  $(\text{card}(A))^{\frac{3}{4}}$ .

Let  $\Psi : R_A \rightarrow Z$  be the map defined by  $\Psi(\delta_j) = y_j$ . We claim that  $\|\Psi\|_{cb} \leq 18$ . Indeed, if  $a_j \in B(\mathcal{H})$  for  $j \in A$ ,

$$\sum_{j \in A} a_j \otimes y_j = \sum_{j \in A} a_j \otimes \sum_{k=1}^9 \lambda_{j,k} e_{f_k(j)} = \sum_{k=1}^9 \left[ \sum_{j \in A} \lambda_{j,k} a_j \otimes e_{f_k(j)} \right].$$

It follows from (3) of Lemma 2 that  $\{f_k(j) : j \in A\} \subset B$  for some  $B$  in  $\Delta_n, \Delta_{n+1}$ , or  $\Delta_{n+2}$ . Then the definition of  $X$  implies that the span of the  $e_{f_k(j)}$ 's for  $j \in A$  is a row operator space. Hence,

$$\left\| \sum_{j \in A} a_j \otimes y_j \right\| \leq (9)(2) \left\| \sum_{j \in A} a_j a_j^* \right\|^{\frac{1}{2}} = 18 \left\| \sum_{j \in A} a_j \otimes \delta_j \right\|.$$

Let  $X_A = \text{span}\{e_j : j \in A\}$ . By (2) of Lemma 2, all the elements of  $A$  belong to different elements of the partition  $\Delta_{n+1}$ . This implies that  $X_A$  is completely isometric to  $OH_A$ . Let  $P_A : X \rightarrow X_A$  be the completely contractive projection onto  $X_A$ , and let  $I : X_A \rightarrow R_A$  be the formal identity. Then we have that

$$\alpha = \Psi \circ I \circ P_A \circ \iota_Z.$$

If  $T : Z^* \rightarrow Z^*$  is completely bounded,

$$\begin{aligned} |\langle T, \alpha \rangle| &= |\text{tr}(T^* \circ \alpha)| = |\text{tr}(T^* \circ \Psi \circ I \circ P_A \circ \iota_Z)| = |\text{tr}(P_A \circ \iota_Z \circ T^* \circ \Psi \circ I)| \\ &\leq \|P_A \circ \iota_Z \circ T^* \circ \Psi\|_{cb} \|I\|_{OH_A \hat{\otimes} R_A} \leq 18 \|T\|_{cb} (\text{card}(A))^{\frac{3}{4}}. \end{aligned}$$

This finishes the proof of Lemma 3.  $\square$

**Remarks.** We can replace the  $\ell_2$  sum of  $R_n$ 's by the  $\ell_2$  sum of  $C_n$ 's or by the  $\ell_2$  sum of any other family of homogeneous Hilbert spaces which is “far” from  $OH$  (e.g.,  $R_p^n = (R_n, C_n)_{\frac{1}{p}}$ ,  $p \neq 2$ ). The same proof gives that all of these spaces have subspaces failing the operator approximation property. We can also replace the  $\ell_2$  sum of  $R_n$ 's by the  $\ell_p$  sum of  $R_n$ 's. However, when  $p \neq 2$ , then  $X$  is no longer a Hilbert space. The subspace  $Z$  of  $X$  will fail the operator approximation property, but Szankowski's Theorem actually gives that  $Z$  already fails the Banach approximation property.

### 3. AN EXAMPLE IN NONCOMMUTATIVE $L_p$

We recall standard facts about noncommutative  $L_p$  spaces.  $S_p$  is the Schatten  $p$ -class on  $\ell_2$  with the operator space structure induced by the complex interpolation  $S_p = (S_\infty, S_1)_{\frac{1}{p}}$  (see [P2]).  $R_p = \overline{\text{span}}\{e_{1n} : n \in \mathbb{N}\}$  is the row of  $S_p$ , and  $C_p = \overline{\text{span}}\{e_{n1} : n \in \mathbb{N}\}$  is the column of  $S_p$ . Both of them are taken with the operator space structure they inherit from  $S_p$ . It follows from Lemma 1.7 of [P2], that the operator space structure of any operator space  $E$  is determined by  $S_p[E]$ , the noncommutative  $E$ -valued  $S_p$ -space. In particular, the operator space structures of  $R_p$  are  $S_p$  are determined by  $S_p[R_p]$  and  $S_p[C_p]$ . That is, if  $a_n \in S_p$  is a finite family,  $\|\sum_n a_n \otimes e_{1n}\|_{S_p[R_p]} = \|(\sum_n a_n a_n^*)^{\frac{1}{2}}\|_{S_p}$  and  $\|\sum_n a_n \otimes e_{n1}\|_{S_p[C_p]} = \|(\sum_n a_n^* a_n)^{\frac{1}{2}}\|_{S_p}$ . We will also consider  $R_p^n$  and  $C_p^n$ , the row and column of  $S_p^n$ ; and more generally, if  $B$  is a subset of  $\mathbb{N}$ ,  $R_p^B$  and  $C_p^B$  are the row and column of  $S_p^B$ , the Schatten  $p$ -class on the Hilbert space with orthonormal basis indexed by  $B$ .

The operator space structure of  $L_p[0, 1]$  is determined by  $S_p[L_p]$ . We note that by Proposition 2.1 of [P2],  $S_p[L_p]$  is completely isometric to the more familiar  $L_p[S_p]$ . We will consider  $\mathcal{R}_p \subset L_p[0, 1]$ , the subspace generated by the Rademacher functions  $(\epsilon_n)_{n \in \mathbb{N}}$ , with the operator space structure inherited from  $L_p$ . It is known that the operator space structure of  $\mathcal{R}_p$  is determined by the noncommutative Khintchine's inequalities of [L-P] and [L-P P]. If  $p \geq 2$ ,  $\mathcal{R}_p$  is completely isomorphic to  $R_p \cap C_p$  and  $\mathcal{R}_p^n$  is completely isomorphic to  $R_p^n \cap C_p^n$ , with a constant depending only on  $p$ . If  $1 \leq p \leq 2$ ,  $\mathcal{R}_p$  is completely isomorphic to  $R_p + C_p$  and  $\mathcal{R}_p^n$  is completely isomorphic to  $R_p^n + C_p^n$ , with a constant depending only on  $p$  (see [P2, Section 8.4] for details).

Recall that the classical Khintchine's inequality states that there exist constants  $A_p$  and  $B_p$  such that for any square summable sequence  $(\alpha_n)$

$$(2) \quad A_p \left\| \sum_n \alpha_n \epsilon_n \right\|_2 \leq \left\| \sum_n \alpha_n \epsilon_n \right\|_p \leq B_p \left\| \sum_n \alpha_n \epsilon_n \right\|_2.$$

We will now construct a Hilbertian subspace of  $L_p[S_p]$  as a combination of Rademacher functions in  $L_p$  and the row of  $S_p$ . The idea is to put a finite set of Rademacher functions in the (1,1) row of  $S_p$ , then to put another finite set of Rademacher functions in the (1,2) row of  $S_p$ , and another finite set of Rademacher functions in the (1,3) row of  $S_p$ , and so on. At the end we get a space  $X$  which plays the same role as the space  $X$  of the previous section. To make the construction more precise, recall that for each  $n \in \mathbb{N}$ ,  $\Delta_n$  is a partition of  $\sigma_n = \{2^n, 2^n + 1, \dots, 2^{n+1}\}$ . Then  $\mathcal{P} = \{B \in \Delta_n : n \in \mathbb{N}\}$  is a partition of  $\mathbb{N}$ . For convenience, we index the row of  $S_p$  by the countable set  $\mathcal{P}$ . That is,  $R_p = \overline{\text{span}}\{e_{1B} : B \in \mathcal{P}\}$ . Define

$$X = \overline{\text{span}}\{e_{1B} \otimes \epsilon_k : B \in \mathcal{P}, k \in B\} \subset S_p[L_p] \equiv L_p[S_p].$$

Alternatively, define  $e_k := e_{1B} \otimes \epsilon_k \in S_p[L_p]$ , where  $B \in \mathcal{P}$  is the element in the partition that contains  $k \in \mathbb{N}$ , and let  $X$  be the closed span of the  $e_k$ 's.

**Proposition 4.** *At the Banach space level,  $X$  is isomorphic to a Hilbert space.*

**Proof.** An element  $f \in X$  has the form  $f = \sum_{B \in \mathcal{P}} e_{1B} \otimes f_B$ , where  $f_B$  belongs to  $\text{span}\{\epsilon_k : k \in B\} = \mathcal{R}_p^B$ . Since  $S_p[L_p]$  is completely isometric to  $L_p[S_p]$ , we view  $f : [0, 1] \rightarrow S_p$  as an  $S_p$ -valued function with norm

$$\|f\|_{L_p[S_p]} = \left( \int_0^1 \|f(t)\|_{S_p}^p dt \right)^{\frac{1}{p}} = \left[ \int_0^1 \left( \sum_{B \in \mathcal{P}} |f_B(t)|^2 \right)^{\frac{p}{2}} dt \right]^{\frac{1}{p}}.$$

If  $p \geq 2$ , then  $\|f\|_{L_p[S_p]} \geq \left( \int_0^1 (\sum_{B \in \mathcal{P}} |f_B(t)|^2)^{\frac{p}{2}} dt \right)^{\frac{1}{p}} = \sqrt{\sum_{B \in \mathcal{P}} \|f_B\|_2^2}$ . To check the opposite direction, we use that  $L_p[S_p]$  has type 2, and that  $\sum_{B \in \mathcal{P}} e_{1B} \otimes f_B$  in an unconditional sum. Then,  $\|f\|_{S_p[L_p]} \leq C \sqrt{\sum_{B \in \mathcal{P}} \|f_B\|_p^2}$ . Since  $f_B \in \mathcal{R}_p^B$ , it follows from (2) that  $\|f_B\|_p \leq B_p \|f_B\|_2$ . Therefore,  $\|f\|_{S_p[L_p]}$  is equivalent to  $\sqrt{\sum_{B \in \mathcal{P}} \|f_B\|_2^2}$ , which is the norm of a Hilbert space.

The case  $1 \leq p \leq 2$  is similar. We first check that  $\|f\|_{L_p[S_p]} \leq \sqrt{\sum_{B \in \mathcal{P}} \|f_B\|_2^2}$ . Then we use the cotype 2 of  $L_p[S_p]$  and the Rademacher estimate of (2) to conclude that  $\|f\|_{L_p[S_p]} \geq C \sqrt{\sum_{B \in \mathcal{P}} \|f_B\|_2^2}$ .  $\square$

The operator space structure of  $X$  can be described easily. The blocks of  $X$  are indexed by  $B \in \mathcal{P}$ , and they are completely isometric to  $\text{span}\{\epsilon_k : k \in B\} = \mathcal{R}_p^B$ . When  $p \geq 2$ ,  $\mathcal{R}_p^B$  is completely isomorphic (with constant depending only on  $p$ ) to  $R_p^B \cap C_p^B$ , and when  $1 \leq p \leq 2$ ,  $\mathcal{R}_p^B$  is completely isomorphic to  $R_p^B + C_p^B$ . Since the sum is taken in an  $R_p$  sense, we obtain the following complete isomorphisms

$$\begin{aligned} X &\approx \left( \sum_{n=1}^{\infty} \sum_{B \in \Delta_n} \oplus [R_p^B \cap C_p^B] \right)_{R_p} && \text{for } p \geq 2, \text{ and} \\ X &\approx \left( \sum_{n=1}^{\infty} \sum_{B \in \Delta_n} \oplus [R_p^B + C_p^B] \right)_{R_p} && \text{for } 1 \leq p \leq 2. \end{aligned}$$

To apply Enflo's criterion, we need the following estimates.

**Lemma 5.** *Let  $1 \leq p \leq \infty$ . Then the normed spaces  $CB(C_p^n, R_p^n)$ ,  $CB(R_p^n, C_p^n)$ ,  $CB(C_p^n, R_p^n \cap C_p^n)$ , and  $CB(R_p^n, R_p^n \cap C_p^n)$  are isomorphic to  $S_r^n$ , the Schatten  $r$ -class, where  $r$  satisfies  $\frac{1}{r} = |\frac{1}{p} - \frac{1}{2}|$ .*

**Proof.** It is enough to prove that  $CB(C_p^n, R_p^n)$  is isomorphic to  $S_r^n$ . The proof for the second space is similar, and the proof for the last two spaces follows from the first two. By duality, it is enough to assume that  $1 \leq p \leq 2$ . Let  $T \in CB(C_p^n, R_p^n)$ . Write  $T = UDV$  where  $U : R_p^n \rightarrow R_p^n$  and  $V : C_p^n \rightarrow C_p^n$  are unitary, and  $D : C_p^n \rightarrow R_p^n$  is the diagonal operator  $D e_{i1} = \lambda_i e_{1i}$  with  $\lambda_i$  equal to the  $i$ th singular number of  $T$ .

Since  $R_p^n$  and  $C_p^n$  are homogeneous Hilbert spaces, we conclude that  $\|T\|_{cb} = \|D\|_{cb}$ . By Lemma 1.7 of [P2],

$$\|D\|_{cb} = \sup \left\{ \frac{\left\| \sum_{i \leq n} \lambda_i e_{1i} \otimes a_i \right\|_{R_p^n[S_p]}}{\left\| \sum_{i \leq n} e_{i1} \otimes a_i \right\|_{C_p^n[S_p]}} : a_i \in S_p, \quad i \leq n \right\}.$$

It is well known that for  $1 \leq p \leq 2$ ,  $\left\| \sum_{i \leq n} \lambda_i e_{1i} \otimes a_i \right\| \leq \left( \sum_{i \leq n} |\lambda_i|^p \|a_i\|_{S_p}^p \right)^{\frac{1}{p}}$ , and  $\left\| \sum_{i \leq n} e_{i1} \otimes a_i \right\| \geq \left( \sum_{i \leq n} \|a_i\|_{S_p}^2 \right)^{\frac{1}{2}}$ . Moreover, if  $a_i = b_i e_{i1} \in S_p^n$  for some scalars  $b_i \in \mathbb{C}$ , the inequalities are attained. Since  $\|a_i\|_{S_p} = |b_i|$ , we have

$$\|D\|_{cb} = \sup \left\{ \left( \sum_{i \leq n} |\lambda_i b_i|^p \right)^{\frac{1}{p}} : \sqrt{\sum_{i \leq n} |b_i|^2} \leq 1 \right\} = \left( \sum_{i \leq n} |\lambda_i|^r \right)^{\frac{1}{r}} = \|T\|_{S_p^n}.$$

This finishes the proof.  $\square$

Since  $CB(C_p^n, R_p^n \cap C_p^n)$  is the dual of  $(R_p^n)^* \hat{\otimes} (R_p^n + C_p^n)$ , it follows from Lemma 5 that for every  $T \in (R_p^n)^* \hat{\otimes} (R_p^n + C_p^n)$ ,  $\|T\|_{(R_p^n)^* \hat{\otimes} (R_p^n + C_p^n)} = \|T\|_{S_p^n}$ , where  $\frac{1}{r} + \frac{1}{r'} = 1$ . In particular, if  $T = I_n$  is the ‘‘formal’’ identity map, then

$$(3) \quad \|I_n\|_{(R_p^n)^* \hat{\otimes} (R_p^n + C_p^n)} = n^{\frac{1}{r'}}.$$

Similarly,

$$(4) \quad \|I_n\|_{(R_p^n \cap C_p^n)^* \hat{\otimes} R_p^n} = n^{\frac{1}{r}}.$$

With these estimates, we can follow Szankowski’s construction to obtain the following Theorem. We sketch the proof:

**Theorem 6.** *For every  $1 \leq p < \infty$ ,  $p \neq 2$ , there exists a Hilbertian subspace of  $L_p[S_p]$  without the operator approximation property.*

Recall that  $X$  is the closed span of  $e_k := e_{1B} \otimes e_k$ ,  $k \in \mathbb{N}$ . For each  $i = 1, 2, \dots$ , let  $z_i = e_{2i} - e_{2i+1} + e_{4i} + e_{4i+1} + e_{4i+2} + e_{4i+3}$  and  $z_i^* = \frac{1}{2}(e_{2i}^* - e_{2i+1}^*)$ , where the  $e_i^*$ ’s are biorthogonal to the  $e_i$ ’s. For each  $n \geq 2$ , let  $\beta_n = \frac{1}{2^n} \sum_{i \in \sigma_n} z_i^* \otimes z_i$ . Finally, let  $Z$  be the closed span of the  $z_i$ ’s. The claim is that there exist partitions  $(\Delta_n)_n$  such that  $Z$  fails of the OAP. We prove this by checking the three conditions of Enflo’s Criterion. The first one is trivial. The second one follows easily. We estimate the cb-norm of  $\beta_n$  through the cb-norm of  $\tilde{\beta}_n$ , which has the same matrix representation of  $\beta_n$  but it is defined on  $R_p$ , the row of  $S_p$ .

We use Szankowski’s partitions of Lemma 2 to check the third condition of Enflo’s Criterion for  $1 \leq p < 2$ . Recall that  $\beta_n - \beta_{n-1} = \frac{1}{2^{n+1}} \sum_{j \in \sigma_{n+1}} e_j^* \otimes y_j = \frac{1}{2^{n+1}} \sum_{A \in \nabla_{n+1}} \left[ \sum_{j \in A} e_j^* \otimes y_j \right]$ , where  $y_j = \sum_{k=1}^9 \lambda_{j,k} e_{f_k(j)} \in Z$  for some  $|\lambda_{j,k}| = 1$  or 2. Let  $A \in \nabla_{n+1}$ . It follows from Lemma 2 (2) that  $X_A = \text{span}\{e_i : i \in A\}$  is completely isomorphic to  $R_p^A$ . And if  $\Psi : R_p^A + C_p^A \rightarrow Z$  is defined by  $\Psi(\delta_j) = y_j$ , it follows from Lemma 2 (3) that  $\|\Psi\|_{cb} \leq 18$ . Note that  $\sum_{j \in A} e_j^* \otimes y_j = \sum_{j \in A} e_j^* \otimes \Psi(\delta_j) = (I \otimes \Psi) \sum_{j \in A} e_j^* \otimes \delta_j$ , and that  $\sum_{j \in A} e_j^* \otimes \delta_j$  is the ‘‘formal’’ identity map on  $(R_p^A)^* \hat{\otimes} (R_p^A + C_p^A)$ . Then it follows from (3) that

$$\left\| \sum_{j \in A} e_j^* \otimes y_j \right\|_{Z^* \hat{\otimes} Z} \leq \|\Psi\|_{cb} \left\| \sum_{j \in A} e_j^* \otimes \delta_j \right\|_{(R_p^A)^* \hat{\otimes} (R_p^A + C_p^A)} \leq 18(\text{card}(A))^{\frac{1}{r}},$$

where  $\frac{1}{r} + \frac{1}{r'} = 1$  and  $\frac{1}{r} = \left| \frac{1}{p} - \frac{1}{2} \right|$ . Now, we easily check  $\sum_n \|\beta_n - \beta_{n-1}\|_{Z^* \hat{\otimes} Z} < \infty$ .

To prove the third condition of Enflo's Criterion for  $p > 2$ , we consider the variation of Lemma 2 described in [LT] page 111. Namely, we find partitions  $\Delta_n$ ,  $\nabla_n$  such that every  $A \in \nabla_n$  is contained in some element of  $\Delta_n$  while, for every  $A \in \nabla_n$ ,  $k = 1, \dots, 9$ , and  $B$  in  $\Delta_{n-1}$ ,  $\Delta_n$ , or  $\Delta_{n+1}$ ,  $\text{card}(B \cap f_k(A)) \leq 1$ . Then, if  $A \in \nabla_{n+1}$ ,  $X_A$  is completely isomorphic to  $R_p^A \cap C_p^A$ , and if  $\Psi : R_p^A \rightarrow Z$  is defined by  $\Psi(\delta_j) = y_j$ , then  $\|\Psi\|_{cb} \leq 18$ . Since  $\sum_{j \in A} e_j^* \otimes \delta_j$  is the "formal" identity map on  $(R_p^A \cap C_p^A)^* \hat{\otimes} R_p^A$ , it follows from (4) that  $\|\sum_{j \in A} e_j^* \otimes y_j\|_{Z^* \hat{\otimes} Z} \leq 18(\text{card}(A))^{\frac{1}{p}}$ , and the third condition of Enflo's Criterion follows.

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ALVARO ARIAS  
 DIVISION OF MATHEMATICS AND STATISTICS, THE UNIVERSITY OF TEXAS AT SAN ANTONIO,  
 SAN ANTONIO, TX 78249, U.S.A.  
*E-mail address:* arias@math.utsa.edu