MULTIPLIERS AND REPRESENTATIONS OF NONCOMMUTATIVE DISC ALGEBRAS

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Abstract. The non-commutative disc algebra A_n, $n \geq 2$, is the norm closure of the non-selfadjoint algebra generated by the left regular representation of $\mathbb{F}_n^+$, the free semigroup on n generators. We present examples of contractive representation of A_n, $n \geq 2$, which are not completely contractive. This answers a question from [Po7]. We characterize the (completely) contractive Schur multipliers of A_n which are indexed by subsets of $\mathbb{F}_n^+$, and we show that for $n \geq 2$, $B(\mathbb{F}_n^+) \subset M_0(\mathbb{F}_n^+)$ and $M_0(\mathbb{F}_n^+) \setminus B(\mathbb{F}_n^+) \neq \emptyset$. $B(\mathbb{F}_n^+)$ is the space of coefficients of contractive representations of $\mathbb{F}_n^+$ and $M_0(\mathbb{F}_n^+)$ is the space of completely bounded Schur multipliers of A_n.

0. Introduction

In this paper we study some properties of the non-commutative disc algebras A_n, $n \geq 2$. These non-selfadjoint algebras were introduced in 1991 by Popescu. In a sequence of papers (see [Po2], [Po3], [Po4], [Po5], [Po6], and [Po7]) he established remarkable similarities between these algebras and classical spaces appearing in Harmonic Analysis. This line of study has been pursued recently by Davidson and Pitts (see [DP1] and [DP2]). We also refer to [A], [APo], and [DPo] for related results.

For the moment, it suffices to say that A_n is an algebra generated by n isometries with orthogonal ranges $S_1, S_2, \cdots, S_n$. The finite products of the S_i's can be indexed naturally by $\mathbb{F}_n^+$, the free semigroup on n generators $g_1, g_2, \cdots, g_n$. That is, if $\alpha = g_{i_1} g_{i_2} \cdots g_{i_k} \in \mathbb{F}_n^+$, then

$$S_\alpha = S_{i_1} S_{i_2} \cdots S_{i_k}, \qquad \text{and} \qquad S_0 = I.$$

Our study is focused on maps $u : \mathbb{F}_n^+ \to \mathbb{C}$. We say that u is a bounded Schur multiplier of A_n (respectively, completely bounded Schur multiplier of A_n) if the operator $\Phi_u : A_n \to A_n$ defined by $\Phi_u(S_\alpha) = u(\alpha) S_\alpha$ is bounded (respectively, completely bounded). The set of bounded Schur multipliers of A_n is denoted by $M(\mathbb{F}_n^+)$, and the set of completely bounded Schur multiplier of A_n is denoted by $M_0(\mathbb{F}_n^+)$. We say that $u : \mathbb{F}_n^+ \to \mathbb{C}$ is a coefficient of a contractive representation of $\mathbb{F}_n^+$ if there exist a Hilbert space H, a contractive representation $\pi : \mathbb{F}_n^+ \to B(H)$, and two vectors $\xi, \eta \in H$ such that for each α, $u(\alpha) = \langle \pi(\alpha)\xi, \eta \rangle$. The set of coefficients of contractive representations is denoted by $B(\mathbb{F}_n^+)$.

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In Section 1 we set the notation and state the necessary background material. In Sections 2 and 3 we study Schur multipliers of A_n indexed by subsets of $\mathbb{F}_n^+$. If $\Lambda \subset \mathbb{F}_n^+$, define $u_\Lambda : \mathbb{F}_n^+ \to \mathbb{C}$ by

$$u_\Lambda(\alpha) = \begin{cases} 1, & \text{if } \alpha \in \Lambda, \\ 0, & \text{if } \alpha \notin \Lambda. \end{cases}$$

In Section 2 we characterize the subsets Λ of $\mathbb{F}_n^+$ such that u_Λ is a completely contractive multiplier, and in Section 3 we show that such multipliers are contractive if and only if they are completely contractive. Some of the results of these sections can be easily extended to other discrete semigroups. In Section 4 we show that $B(\mathbb{F}_n^+) \subset M_0(\mathbb{F}_n^+)$ and that $M_0(\mathbb{F}_n^+) \setminus B(\mathbb{F}_n^+) \neq \emptyset$ for $n \geq 2$. We also establish some properties of these spaces. Finally, in Section 5, we present an example of a contractive representation of A_n, $n \geq 2$, that is not completely contractive. This answers a question from [Po7]. We also give a simple example of a bounded representation of A_n, $n \geq 2$, that is not completely bounded.

Several questions addressed in this paper are motivated by results on Herz-Schur multipliers of the Fourier algebra of locally compact groups (see [CoH], [DCH], [BF], [Ne]). If G is a discrete group, $C_\lambda^*(G)$ is the C^*-algebra of the left regular representation of G, $M(G)$ is the space of bounded Schur multipliers of $C_\lambda^*(G)$, $M_0(G)$ is the space of completely bounded Schur multipliers of $C_\lambda^*(G)$, and $B(G)$ is the space of coefficients of unitary representations of G. We refer the reader to [E], [Gr], [Pat], and [Pi] for more information. It is well known that

$$B(G) \subset M_0(G) \subset M(G).$$

Moreover, $B(G) = M_0(G)$ if and only if G is amenable (see [BF], [Ne], and [Wy]). Since $\mathbb{F}_\infty$, the free group on countably many generators, is not amenable, then $M_0(\mathbb{F}_\infty) \setminus B(\mathbb{F}_\infty) \neq \emptyset$. One can show that the function

$$u : \mathbb{F}_\infty \to \mathbb{C} \quad \text{defined by} \quad u(\alpha) = \begin{cases} 1, & \text{if } |\alpha| = 1, \\ 0, & \text{otherwise,} \end{cases}$$

belongs to $M_0(\mathbb{F}_\infty) \setminus B(\mathbb{F}_\infty)$ (see [Wy] and [P, Lemma 2.2]).

1. PRELIMINARIES

The non-commutative disc algebras A_n, $n \geq 2$, are subalgebras of $B(\ell_2(\mathbb{F}_n^+))$, where $\mathbb{F}_n^+$ is the free semigroup on n generators $g_1, \cdots, g_n$, and $\ell_2(\mathbb{F}_n^+)$ is the Hilbert space with orthonormal basis $\{\delta_\alpha : \alpha \in \mathbb{F}_n^+\}$. The left regular representation $\lambda : \mathbb{F}_n^+ \to B(\ell_2(\mathbb{F}_n^+))$ is defined by $\lambda(\alpha)\delta_\beta = \delta_{\alpha\beta}$. The non-commutative disc algebra A_n is the norm closure of $\operatorname{span}\{\lambda(\alpha) : \alpha \in \mathbb{F}_n^+\}$ in $B(\ell_2(\mathbb{F}_n^+))$.

An alternative description of A_n uses non-commutative polynomials. Let $\mathcal{P} = \mathcal{P}_n$ be the set of polynomials in n non-commutative variables $e_1, \cdots, e_n$, with product $\otimes$ and identity e_0. A typical element of $\mathcal{P}$ looks like

$$p = a_0 e_0 + \sum_{i=1}^{N} \sum_{i_1, i_2, \cdots i_k = 1}^{n} a_{i_1 i_2 \cdots i_k} e_{i_1} \otimes e_{i_2} \otimes \cdots e_{i_k}.$$

The Hilbert space with orthonormal basis $\{e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_k}\}$ is called the full Fock space of the n-dimensional Hilbert space H_n with orthonormal basis $e_1, \cdots e_n$. This space is usually denoted by

$$\mathcal{F}^2(H_n) = \sum_{k=0}^{\infty} \bigoplus \ H_n^{\otimes k},$$

where $H_n^{\otimes 0} = \mathbb{C}e_0$ and $H_n^{\otimes k}$ is the tensor product of k copies of H_n. The basis of $\mathcal{F}^2(H_n)$ can be indexed by $\mathbb{F}_n^+$. If $\alpha = g_{i_1} g_{i_2} \cdots g_{i_k} \in \mathbb{F}_n^+$, let $e_\alpha = e_{i_1} \otimes e_{i_2} \otimes \cdots e_{i_k}$. Then $\{e_\alpha : \alpha \in \mathbb{F}_n^+\}$ is the canonical orthonormal basis of $\mathcal{F}^2(H_n)$.

We can view the elements of $\mathcal{P}$ as left multiplication operators on $\mathcal{F}^2(H_n)$. If $p \in \mathcal{P}$, the map $p : \mathcal{F}^2(H_n) \to \mathcal{F}^2(H_n)$, defined by $\varphi \to p \otimes \varphi$, has norm

$$\|p\| = \sup\{\|p \otimes \varphi\|_2 : \|\varphi\|_2 = 1, \varphi \in \mathcal{F}^2(H_n)\}.$$

The non-commutative disc algebra A_n is the norm-closure of $\mathcal{P}$ in $B(\mathcal{F}^2(H_n))$. When $n = 1$, A_n coincides with the disc algebra A.

The two descriptions of A_n are equivalent. To see this, identify δ_β with e_β, and $\lambda(\alpha)$ with e_α. Then, $\ell_2(\mathbb{F}_n^+)$ is canonically isomorphic to $\mathcal{F}^2(H_n)$, and the equality $\lambda(\alpha)\delta_\beta = \delta_{\alpha\beta}$, which describes $\lambda(\alpha)$, is transformed into $e_\alpha \otimes e_\beta = e_{\alpha\beta}$, which describes e_α as a left multiplication operator on $\mathcal{F}^2(H_n)$.

Tuples of isometries $S_1, S_2, \cdots, S_n$ with orthogonal ranges have been studied for some time. Particularly in Cuntz' algebras (see [C]) and in Dilation Theory (see [Bu], [F1], [F2], [Po1], [Po2], and [Po6]). In [Po2, Theorem 2.2], Popescu gave a Beurling-Lax characterization of the invariant subspaces of $S_1, S_2, \cdots, S_n$. In 1989, he used this description to obtain a very general "inner-outer" factorization of operators (see [Po3, Theorem 4.2]). In 1991, he introduced the algebras A_n, $n \geq 2$, and their wot-closure, $\mathcal{F}^\infty(H_n)$, $n \geq 2$ (see [Po4]). The factorization result from [Po3] is very transparent for these algebras. Popescu said that $\varphi \in \mathcal{F}^\infty(H_n)$ is inner if φ is an isometry, and $\psi \in \mathcal{F}^\infty(H_n)$ is outer if ψ has dense range (recall that φ and ψ are operators on $\ell_2(\mathbb{F}_n^+)$). Then, Theorem 4.2 of [Po3] says that any $\eta \in \mathcal{F}^\infty(H_n)$ can be written as $\eta = \varphi \otimes \psi$ were φ is inner and ψ is outer. Moreover, this factorization is essentially unique.

Inner operators are useful to study A_n and $\mathcal{F}^\infty(H_n)$. For example, they were used in [APo] to show that $\mathcal{F}^\infty(H_n)$ is a reflexive algebra and in [DP1] to show that $\mathcal{F}^\infty(H_n)$ is hyper-reflexive. In this paper we will use the following three examples, which appeared in [APo]: *(i) Inherited inner functions*: Let $\varphi(z) \in A$ and $\alpha \in \mathbb{F}_n^+$, then $\varphi(e_\alpha) \in A_n$ and $\|\varphi\|_A = \|\varphi(e_\alpha)\|_{A_n}$. If φ is inner in A, then $\varphi(e_\alpha)$ is inner in A_n. *(ii) Homogeneous polynomials*: Let $p \in A_n$ be a homogeneous polynomial of degree k, (i.e., $p \in \operatorname{span}\{e_\alpha : |\alpha| = k\}$, where $|\alpha|$ is the length of the word α), then $\|p\|_{A_n} = \|p\|_2$. If $\|p\|_2 = 1$, then p is inner. *(iii)* Let $p(z)$ be a polynomial in the disc algebra A, then $\|p(e_1) \otimes e_2\|_{A_n} = \|p(e_1) \otimes e_2\|_2$. If $\|p\|_2 = 1$, then $p(e_1) \otimes e_2$ is inner. We refer to [A] and [DP1] for related results.

Two words $\alpha, \beta \in \mathbb{F}_n^+$ are said to be orthogonal if $\alpha\mathbb{F}_n^+ \bigcap \beta\mathbb{F}_n^+ = \emptyset$. This is equivalent to saying that $\lambda(\alpha)$ and $\lambda(\beta)$ have orthogonal ranges. Or, that for every $\varphi_1, \varphi_2 \in \ell_2(\mathbb{F}_n^+)$, $e_\alpha \otimes \varphi_1$ is orthogonal to $e_\beta \otimes \varphi_2$.

The unitary "flip" $\Theta : \ell_2(\mathbb{F}_n^+) \to \ell_2(\mathbb{F}_n^+)$ is defined by

$$\Theta(e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_k}) = e_{i_k} \otimes \cdots \otimes e_{i_2} \otimes e_{i_1}.$$

Its action is denoted by $\Theta(\varphi) = \tilde{\varphi}$. This map is unbounded on A_n.

An *operator space* X is a subspace of $B(H)$. The *minimal* (or spatial) tensor product of two operator spaces X and Y is denoted by $X \otimes_{min} Y$. If $X \subset B(H)$ and $Y \subset B(K)$, this is the just the closure of the algebraic tensor product $X \otimes Y$ in $B(H \otimes_2 K)$, where $H \otimes_2 K$ is the Hilbert-Schmidt tensor norm of H and K. Since $M_n \otimes_{min} B(H) \equiv M_n(B(H))$, one identifies $M_n(X)$ with $M_n \otimes_{min} X$. Thus, an operator space X has a natural family of norms $\|\cdot\|_n$ on $M_n(X)$.

The morphisms in the category of operator spaces are the *completely bounded maps*. A bounded linear map $\Phi : X \to Y$ between two operator spaces X and Y is *completely bounded*, or cb, if the map $id_Z \otimes \Phi : Z \otimes_{min} X \to Z \otimes_{min} Y$ is bounded for every operator space Z. The completely bounded norm $\|\Phi\|_{cb}$ of Φ is the supremum of the norms $id_Z \otimes \Phi$ as Z ranges over all operator spaces. It turns out that it is enough to look only at the M_n's. That is, $\|\Phi\|_{cb} = \sup_n \|\Phi_n\|$, where $\Phi_n = id_n \otimes \Phi : M_n(X) \to M_n(Y)$.

The fundamental theorem of completely bounded maps is due to Wittstock [W]. It was discovered independently by Haagerup and Paulsen, and the factorization was inspired by results of Stinespring [S] and Arveson [Arv].

Theorem 1.1 [W]. *Let $X \subset B(H)$ be an operator space, $\Phi : X \to B(\ell_2)$ a bounded linear map, and $C > 0$. Φ is completely bounded and $\|\Phi\|_{cb} \le C$ if and only if there exist a $*$-representation $\pi : B(H) \to B(\ell_2)$ and two bounded linear maps $V_1, V_2 : \ell_2 \to H$, $\|V_1\|\|V_2\| \le C$, such that for every $x \in X$, $\Phi(x) = V_2^* \pi(x) V_1$.*

If $\Phi : A_n \to A_n$ is a completely bounded Schur multiplier of A_n, then there exist a Hilbert space H, a $*$-representation $\pi : B(\ell_2(\mathbb{F}_n^+)) \to B(H)$, and two bounded linear maps $V_1, V_2 : \ell_2(\mathbb{F}_n^+) \to H$ satisfying $\|V_1\| = \|V_2\|$ and $\|V_1\|\|V_2\| = \|\Phi\|_{cb}$, such that for each $\alpha \in \mathbb{F}_n^+$, $\Phi(\lambda(\alpha)) = V_2^* \pi(\lambda(\alpha)) V_1$. For every $\beta \in \mathbb{F}_n^+$,

$$u(\alpha) = \langle \Phi(\lambda(\alpha))\delta_\beta, \delta_{\alpha\beta}\rangle = \langle V_2^* \pi(\lambda(\alpha)) V_1 \delta_\beta, \delta_{\alpha\beta}\rangle = \langle \pi(\lambda(\alpha)) V_1 \delta_\beta, V_2 \delta_{\alpha\beta}\rangle. \tag{1}$$

We say that a function $u : \Gamma \times \Gamma \to \mathbb{C}$ is a Schur multiplier of $B(\ell_2(\Gamma))$ if the map $M_u : B(\ell_2(\Gamma)) \to B(\ell_2(\Gamma))$, which is defined by $\langle M_u(T)\delta_s, \delta_t\rangle = u(s,t)\langle T\delta_s, \delta_t\rangle$, is bounded. These maps have a nice characterization. The following fundamental result is well known. It is essentially due to Grothendieck [G].

Theorem 1.2. *Let $u : \Gamma \times \Gamma \to \mathbb{C}$ be a function and $C > 0$. The following conditions are equivalent.*

1. *M_u is a bounded Schur multiplier of $B(\ell_2(\Gamma))$ and $\|M_u\| \le C$.*
2. *There exist a Hilbert space H and maps $x, y : \Gamma \to H$ such that $u(s,t) = \langle x(s), y(t)\rangle$ and $\sup_s \|x(s)\| \sup_t \|y(t)\| \le C$.*
3. *M_u is a bounded Schur multiplier of $B(\ell_2(\Gamma))$ and $\|M_u\|_{cb} \le C$.*

If G is a countable group, Bozejko and Fendler [BF] proved that Φ is a completely bounded Schur multiplier of $C^*_\lambda(G)$ if and only if Φ is the restriction of a Schur multiplier M of $B(\ell_2(G))$ with the same norm; i.e., $\|M\| = \|\Phi\|_{cb}$. A simple proof of this fact appears in [P1, Theorem 6.4]. G. Popescu observed that the same argument works in A_n. The following proof is due to him

Proposition 1.3. *Let Φ be a completely bounded Schur multiplier of A_n. Then Φ is the restriction of a Schur multiplier M of $B(\ell_2(\mathbb{F}_n^+))$. Moreover, $\|M\| = \|\Phi\|_{cb}$.*

Proof. Find H, $\pi : B(\ell_2(\mathbb{F}_n^+)) \to B(H)$, and $V_1, V_2 : \ell_2(\mathbb{F}_n^+) \to H$ satisfying (1). Assume that H is infinite dimensional and use the Dilation Theorem to find unitary maps $U_1, \cdots, U_n \in B(H \oplus H \oplus H)$ such that, for each $i \le n$, U_i is upper triangular and the (2,2)-component of U_i is $S_i = \pi(\lambda(g_i))$. Define $\bar{V}_1, \bar{V}_2 : \ell_2(\mathbb{F}_n^+) \to H$ by $\bar{V}_i\varphi = (0, \varphi, 0)$, $i = 1, 2$. Then $\langle \Phi(\lambda(\alpha))\delta_\alpha, \delta_{\alpha\beta}\rangle = \langle S_\alpha V_1\delta_\beta, V_2\delta_{\alpha\beta}\rangle = \langle U_\alpha \bar{V}_1\delta_\beta, \bar{V}_2\delta_{\alpha\beta}\rangle$. Since the U_α's are unitary, $\langle U_\alpha \bar{V}_1\delta_\beta, \bar{V}_2\delta_{\alpha\beta}\rangle = \langle U_\beta^* \bar{V}_1\delta_\beta, U_{\alpha\beta}^* \bar{V}_2\delta_{\alpha\beta}\rangle = \langle x(\beta), y(\alpha\beta)\rangle$, where $x, y : \mathbb{F}_n^+ \to H \oplus H \oplus H$ are defined by $x(\beta) = U_\beta^* \bar{V}_1\delta_\beta$ and $y(\alpha) = U_\alpha^* \bar{V}_2\delta_\alpha$. Therefore, Φ is the restriction of the Schur multiplier M of $B(\ell_2(\mathbb{F}_n^+))$ defined by $\langle MT\delta_\beta, \delta_\alpha\rangle = \langle x(\beta), y(\alpha)\rangle\langle T\delta_\beta, \delta_\alpha\rangle$. Clearly, $\|M\| \le \|V_1\|\|V_2\| = \|\Phi\|_{cb}$. □

2. Completely Contractive Multipliers indexed by subsets of $\mathbb{F}_n^+$

In this section we consider multipliers indexed by subsets of $\mathbb{F}_n^+$. Let $\Lambda \subset \mathbb{F}_n^+$ and define the Λ-multiplier Φ_Λ of A_n by

$$\Phi_\Lambda(\lambda(\alpha)) = \begin{cases} \lambda(\alpha), & \text{if } \alpha \in \Lambda, \\ 0, & \text{otherwise.} \end{cases}$$

The main result is the following:

Theorem 2.1. *The Λ-multiplier Φ_Λ is completely contractive (i.e., $\|\Phi_\Lambda\|_{cb} = 1$) if and only if $\Lambda = \bigcup_k \gamma_k\Lambda_1$ for some semigroup Λ_1 with the left and right cancellation property, and some orthogonal words γ_k with no final segment in Λ_1; i.e., if $\gamma_k = \alpha\beta$ and $\beta \ne 0$, then $\beta \notin \Lambda_1$.*

Suppose that Φ_Λ is a completely contractive multiplier. Find H, $\pi : B(\ell_2(\mathbb{F}_n^+)) \to B(H)$, and $V_1, V_2 : \ell_2(\mathbb{F}_n^+) \to H$ satisfying (1), and note that $\|V_1\| = \|V_2\| = 1$. For every $\alpha \in \Lambda$ and $\beta \in \mathbb{F}_n^+$, $V_2^*\pi(\lambda(\alpha))V_1\delta_\beta = \delta_{\alpha\beta}$. Since $\|V_2^*\pi(\lambda(\alpha))\| \le 1$, we see that $\|V_1\delta_\beta\|_2 = 1$. Hence, V_1 is an isometry. If in addition we assume that $0 \in \Lambda$, we have that $V_1\delta_\beta = \pi(\lambda(0))V_1\delta_\beta = V_2\delta_{0\beta} = V_2\delta_\beta$. Therefore, $V_1 = V_2$.

Proposition 2.2. *If Φ_Λ is completely contractive multiplier and $0 \in \Lambda$, then Λ is a semigroup with the left and right cancellation property.*

Proof. From the comments before the statement of this Proposition, we see that the V_1 and V_2 of (1) are equal to the isometry $V : \ell_2(\mathbb{F}_n^+) \to H$. This provides the following criterion to determine if an element belongs to Λ:

$$\alpha \in \Lambda \iff V\delta_\alpha = \pi(\lambda(\alpha))V\delta_0. \tag{2}$$

We will check first that Λ is closed under products. Let $\alpha, \beta \in \Lambda$. Since $V^*\pi(\lambda(\alpha))V\delta_\beta = \Phi(\lambda(\alpha))\delta_\beta = \lambda(\alpha)\delta_\beta = \delta_{\alpha\beta}$, then

$$V\delta_{\alpha\beta} = \pi(\lambda(\alpha))V\delta_\beta = \pi(\lambda(\alpha))\pi(\lambda(\beta))V\delta_0 = \pi(\lambda(\alpha\beta))V\delta_0.$$

Hence, it follows from (2) that $\alpha\beta \in \Lambda$.

We will now verify that Λ has the left cancellation property. Suppose that $\alpha\beta \in \Lambda$ and $\alpha \in \Lambda$. Then $V^*\pi(\lambda(\alpha))V\delta_\beta = \delta_{\alpha\beta}$. From here we see that $\pi(\lambda(\alpha))(V\delta_\beta) = V\delta_{\alpha\beta} = \pi(\lambda(\alpha\beta))V\delta_0 = \pi(\lambda(\alpha))(\pi(\lambda(\beta))V\delta_0)$. Since $\pi(\lambda(\alpha))$ is an isometry, we conclude that $V\delta_\beta = \pi(\lambda(\beta))V\delta_0$. Therefore, by (2), $\beta \in \Lambda$.

Finally, we will check that Λ has the right cancellation property. Suppose that $\alpha\beta \in \Lambda$ and $\beta \in \Lambda$. Notice that

$$(\Phi(\lambda(\alpha))\delta_\beta, \delta_{\alpha\beta}) = (\pi(\lambda(\alpha))V\delta_\beta, V\delta_{\alpha\beta}) = (\pi(\lambda(\alpha))\pi(\lambda(\beta))V\delta_0, \pi(\lambda(\alpha\beta))V\delta_0) = 1.$$

Hence, $\Phi(\lambda(\alpha)) \neq 0$, so $\alpha \in \Lambda$. □

We will now prove that multipliers indexed by semigroups with the left and right cancellation property are completely bounded. More precisely, we will show that if $\Lambda \subset \mathbb{F}_n^+$ is a semigroup with the left and right cancellation property, then Φ_Λ is the restriction of a completely contractive Schur multiplier of $B(\ell_2(\mathbb{F}_n^2))$. We need the following

Lemma 2.3. *Let Λ be a semigroup with the left and right cancellation property. Define $\alpha \sim \beta$ if there exist $\alpha_1, \alpha_2 \in \Lambda$ and $\gamma \in \mathbb{F}_n^+$ such that $\alpha = \alpha_1\gamma$ and $\beta = \alpha_2\gamma$. Then the relation $\sim$ is an equivalence relation and $\alpha\beta \sim \beta$ if and only if $\alpha \in \Lambda$.*

Proof. Clearly, $\sim$ is reflexive and symmetric. Suppose that $\alpha \sim \beta$ and $\beta \sim \gamma$. Then there exist $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \Lambda$ and $\eta_1, \eta_2 \in \mathbb{F}_n^+$ such that $\alpha = \alpha_1\eta_1$, $\beta = \alpha_2\eta_1$, $\beta = \alpha_3\eta_2$, and $\gamma = \alpha_4\eta_2$. For simplicity, assume that the length of α_2 is greater than or equal to the length of α_3. Then $\alpha_2 = \alpha_3\theta$ for some θ. This implies that $\theta\eta_1 = \eta_2$. Since $\alpha_2, \alpha_3 \in \Lambda$ and Λ has the left cancellation property, $\theta \in \Lambda$. Hence, $\gamma = \alpha_4\eta_2 = (\alpha_4\theta)\eta_1$. Since $(\alpha_4\theta) \in \Lambda$, we get that $\alpha \sim \gamma$. Therefore, $\sim$ is an equivalence relation.

We will now check that $\alpha\beta \sim \beta$ if and only if $\alpha \in \Lambda$. If $\alpha \in \Lambda$, then $\beta \sim \alpha\beta$ for each β. On the other hand, if $\beta \sim \alpha\beta$, there exist $\alpha_1, \alpha_2 \in \Lambda$ and $\gamma \in \mathbb{F}_n^+$ such that $\beta = \alpha_1\gamma$ and $\alpha\beta = \alpha_2\gamma$. Hence, $\alpha\alpha_1 = \alpha_2$. Since Λ has the right cancellation property, we conclude that $\alpha \in \Lambda$. □

Proposition 2.4. *Let $\Lambda \subset \mathbb{F}_n^+$ be a semigroup with the left and right cancellation property. Then there exists a completely contractive Schur multiplier M on $B(\ell_2(\mathbb{F}_n^+))$ such that M restricted to A_n is Φ_Λ.*

Proof. The equivalence classes induced by the relation of Lemma 2.3 are of the form $\Lambda\gamma$, for some $\gamma \in \mathbb{F}_n^+$ with no non-trivial initial segment in Λ. Partition $\mathbb{F}_n^+ = \bigcup_k \Lambda\gamma_k$ and define

$$x_\Lambda : \mathbb{F}_n^+ \to \ell_2 \quad \text{by} \quad x_\Lambda(\alpha) = e_k \quad \text{if} \quad \alpha \in \Lambda\gamma_k.$$

Let M be the completely contractive Schur multiplier of $B(\ell_2(\mathbb{F}_n^+))$ defined by

$$\langle MT\delta_\alpha, \delta_\beta\rangle = \langle x_\Lambda(\alpha), x_\Lambda(\beta)\rangle\langle T\delta_\alpha, \delta_\beta\rangle.$$

From Lemma 2.3, we see that for every $\beta \in \mathbb{F}_n^+$,

$$\langle M(\lambda(\alpha))\delta_\beta, \delta_{\alpha\beta}\rangle = \langle x_\Lambda(\beta), x_\Lambda(\alpha\beta)\rangle = \begin{cases} 1, & \text{if } \beta \sim \alpha\beta, \\ 0, & \text{otherwise.} \end{cases} = \begin{cases} 1, & \text{if } \alpha \in \Lambda, \\ 0, & \text{if } \alpha \notin \Lambda. \end{cases}$$

Therefore, $M\lambda(\alpha) = \Phi_\Lambda(\lambda(\alpha))$. □

Notice that $MT = \sum_k P_k T P_k$, where P_k is the orthogonal projection onto $\ell_2(\Lambda\gamma_k)$. Hence, M is completely positive and $M(\lambda(\alpha)^*) = (M(\lambda(\alpha))^*$ for every $\alpha \in \mathbb{F}_n^+$. As an immediate consequence, we get

Lemma 2.5. *Let Λ be a semigroup with the left and right cancellation property, and $\beta \in \mathbb{F}_n^+$. Let $M = M_\Lambda$ be the Schur multiplier of $B(\ell_2(\mathbb{F}_n^+))$ described in Proposition 2.4. Then $M(\lambda(\beta)^*) \neq 0$ if and only if $\beta \in \Lambda$.*

We will now consider "shifts" of semigroups. We need the following results and examples.

Example 2.6. *It is well known that the multiplier $p(z) \mapsto p(z) - p(0)$ has norm strictly greater than 1 in the disc algebra A.*

Notice that if $\gamma, \alpha \in \mathbb{F}_n^+$, then

$$\lambda(\gamma)^*\lambda(\alpha) = \begin{cases} \lambda(\beta), & \text{if } \alpha = \gamma\beta, \\ \lambda(\beta)^*, & \text{if } \gamma = \alpha\beta, \\ 0, & \text{otherwise.} \end{cases}$$

Proposition 2.7. *Let Λ_1 be a semigroup of $\mathbb{F}_n^+$ with the left and right cancellation property, $\gamma \notin \Lambda_1$, and let $\Lambda = \gamma\Lambda_1$. Then Φ_Λ is completely contractive if and only if no final segment of γ belongs to Λ_1; i.e., if $\gamma = \alpha\beta$ and $\beta \neq 0$, then $\beta \notin \Lambda_1$.*

Proof. Suppose that a non-trivial final segment of γ belongs to Λ_1. Then $\gamma = \alpha\beta$ for some non-zero α, β with $\beta \in \Lambda_1$. This implies that $\beta^k \in \Lambda_1$ for each $k \geq 0$. Since $\gamma \notin \Lambda_1$, then $\alpha \notin \Lambda_1$ either. Hence, $\alpha\beta^k \in \gamma\Lambda_1 = \Lambda$ iff $k \geq 1$. Notice that for any polynomial $p(e_\beta)$ on e_β,

$$\Phi_\Lambda(e_\alpha \otimes p(e_\beta)) = e_\alpha \otimes \Big[p(e_\beta) - p(0)e_0\Big].$$

Using Example 2.6, we get that $\|\Phi_\Lambda\| > 1$.

Suppose now that no final segment of γ belongs to Λ_1. Let M_1 be the completely contractive Schur multiplier of $B(\ell_2(\mathbb{F}_n^+))$ associated to Λ_1 which was described in Proposition 2.4. Define $\Phi = \lambda(\gamma) \circ M_1 \circ (\lambda(\gamma))^*$; i.e., $\Phi(T) = \lambda(\gamma)M_1(\lambda(\gamma)^*T)$. Clearly, Φ is a completely contractive map on $B(\ell_2(\mathbb{F}_n^+))$ and

$$\Phi(\lambda(\alpha)) = \begin{cases} \lambda(\alpha), & \text{if } \alpha \in \gamma\Lambda_1, \\ \lambda(\gamma)M_1(\lambda(\beta)^*), & \text{if } \gamma = \alpha\beta, \\ 0, & \text{otherwise.} \end{cases}$$

If $\gamma = \alpha\beta$ and $\beta \neq 0$ then, by assumption, $\beta \notin \Lambda_1$. Hence, by Lemma 2.5, $M_1(\lambda(\beta)^*) = 0$. Therefore, $\Phi = \Phi_\Lambda$. $\square$

We use the tensor notation for the proof of the following statement. Recall that $\alpha, \beta \in \mathbb{F}_n^+$ are orthogonal words if $e_\alpha \otimes \varphi_1$ is orthogonal to $e_\beta \otimes \varphi_2$ for every φ_1, φ_2 in $\ell_2(\mathbb{F}_n^+)$.

Example 2.8. *Let $\Lambda = \alpha \cup \beta\langle\gamma\rangle$, where α, β are orthogonal words, $\gamma \neq 0$, and $\langle\gamma\rangle$ is the semigroup generated by γ. Then $\|\Phi_\Lambda\| > 1$.*

Proof. Find polynomials $p(z), q(z) \in A$ satisfying $\|p\|_\infty = \|q\|_\infty = 1$, $|p(0)| \geq \frac{1}{2}$, and $\|p \pm q\|_\infty \leq 1 + \epsilon$, where $\epsilon > 0$ is small enough (you can take $p(z) = \frac{1-z}{2}$ and $q(z) = \frac{1}{N}\sum_{k=1}^{N} z^k$, where N is large enough). Notice that

$$\Phi_\Lambda\left[e_\alpha \otimes p(e_\gamma) + e_\beta \otimes q(e_\gamma)\right] = p(0)e_\alpha + e_\beta \otimes q(e_\gamma).$$

It is easy to check that

$$\|p(0)e_\alpha + e_\beta \otimes q(e_\gamma)\| = \sqrt{|p(0)|^2 + \|q\|_\infty^2} \geq \sqrt{\frac{1}{4} + 1} = \sqrt{\frac{5}{4}}.$$

On the other hand, since α and β are orthogonal,

$$\begin{aligned}\|e_\alpha \otimes p(e_\gamma) + e_\beta \otimes q(e_\gamma)\|^2 &= \sup_{\|r\|_2 \leq 1} \|p(e_\gamma) \otimes r\|_2^2 + \|q(e_\gamma) \otimes r\|_2^2 \\ &= \sup_{\|r\|_2 \leq 1} \frac{\|(p+q)(e_\gamma) \otimes r\|_2^2 + \|(p-q)(e_\gamma) \otimes r\|_2^2}{2} \\ &\leq \frac{\|p(e_\gamma) + q(e_\gamma)\|_{A_n}^2 + \|p(e_\gamma) - q(e_\gamma)\|_{A_n}^2}{2} \\ &= \frac{\|p+q\|_\infty^2 + \|p-q\|_\infty^2}{2} \leq (1+\epsilon)^2.\end{aligned}$$

If $\epsilon < \frac{1}{9}$, we get that $\|\Phi_\Lambda\| > 1$. □

Proof of Theorem 2.1. Suppose that Φ_Λ is completely contractive. If $0 \in \Lambda$, it follows from Proposition 2.2 that Λ is a semigroup with the left and right cancellation property. Assume then that $0 \notin \Lambda$ and let B be the set of those $\alpha \in \Lambda$ such that either α has no non-trivial initial segment in Λ, or that α has length one (i.e., $\alpha = g_i$ for some $i \leq n$). Then $B = \{\gamma_1, \gamma_2, \cdots\}$ for some $\gamma_k \in \Lambda$. It is easy to check that the γ_k's are orthogonal words, and that every $\alpha \in \Lambda$ can be written as a product $\alpha = \gamma_k\theta$ for some k and some $\theta \in \mathbb{F}_n^+$.

Let $\gamma_k \in B$ and consider the completely contractive map $\Phi_k : A_n \to B(\ell_2(\mathbb{F}_n^+))$ defined by $\Phi_k = (\lambda(\gamma_k))^* \circ \Phi_\Lambda \circ \lambda(\gamma_k)$. Since

$$\Phi_k(\lambda(\alpha)) = (\lambda(\gamma_k))^* \circ \Phi_\Lambda(\lambda(\gamma_k\alpha)) = \begin{cases} \lambda(\alpha), & \text{if } \gamma_k\alpha \in \Lambda, \\ 0, & \text{otherwise,} \end{cases}$$

we have that $\Phi_k : A_n \to A_n$ is a completely contractive multiplier. Since $\Phi_k(\lambda(0)) = \lambda(0)$, it follows from Proposition 2.2 that $\Phi_k = \Phi_{\Lambda_k}$ for some semigroup Λ_k with the left and right cancellation property. Hence, $\alpha \in \Lambda_k$ if and only if $\gamma_k\alpha \in \Lambda$. This implies that $\Lambda = \bigcup_k \gamma_k\Lambda_k$. We claim that all the Λ_k's are equal to each other. Indeed, if this were not the case, we could find two indices k_1, k_2 and $\gamma \in \mathbb{F}_n^+$ such that $\gamma \notin \Lambda_{k_1}$ and $\gamma \in \Lambda_{k_2}$. Using the notation of Example 2.8, we would get that $\Phi_\Lambda\left(e_{\gamma_{k_1}} \otimes p(e_\gamma) + e_{\gamma_{k_2}} \otimes q(e_\gamma)\right) = p(0)e_{\gamma_{k_1}} + e_{\gamma_{k_2}} \otimes q(e_\gamma)$, which contradicts that Φ_Λ is contractive. Therefore, all the Λ_k's are equal to each other and $\Lambda = \bigcup_k \gamma_k\Lambda_1$.

Suppose now that $\Lambda = \bigcup_k \gamma_k \Lambda_1$, where Λ_1 is a semigroup with the left and right cancellation property, and the γ_k's are orthogonal words with the property that no final segment of γ_k belongs to Λ_1. Let $x_{\Lambda_1} : \mathbb{F}_n^+ \to \ell_2$ be the function associated to Λ_1 which was described in Proposition 2.4.

Define $x, y : \mathbb{F}_n^+ \to \ell_2$ as follows:

$$x(\alpha) = x_{\Lambda_1}(\alpha), \ \forall \alpha \in \mathbb{F}_n^+, \qquad \text{and} \qquad y(\beta) = \begin{cases} x_{\Lambda_1}(\eta), & \text{if } \beta = \gamma_k \eta \text{ for some } k, \\ 0, & \text{otherwise.} \end{cases}$$

Define the completely contractive Schur multiplier of $B(\ell_2(\mathbb{F}_n^+))$ by

$$\langle MT\delta_\alpha, \delta_\beta \rangle = \langle x(\alpha), y(\beta) \rangle \langle T\delta_\alpha, \delta_\beta \rangle.$$

It is easy to check that $M(T) = \sum_k [\lambda(\gamma_k) \circ M_1 \circ \lambda(\gamma_k)^*](T) = \sum_k \lambda(\gamma_k) M_1(\lambda(\gamma_k)^* T)$, where M_1 is the Schur multiplier associate to Λ_1 which was described in Proposition 2.4. Since no final segment of γ_k belongs to Λ_1, it follows from the proof of Lemma 2.5 that

$$\big[\lambda(\gamma_k) \circ M_1 \circ \lambda(\gamma_k)^*\big]\lambda(\alpha) = \begin{cases} \lambda(\alpha), & \text{if } \alpha \in \gamma_k \Lambda_k, \\ 0, & \text{otherwise.} \end{cases}$$

Therefore the restriction of M to A_n is equal to Φ_Λ. □

3. CONTRACTIVE MULTIPLIERS INDEXED BY SUBSETS OF $\mathbb{F}_n^+$

In this Section we will show that a Schur multiplier indexed by a subset of $\mathbb{F}_n^+$ is contractive if and only if it is completely contractive. The main step is Theorem 3.1, which replaces Proposition 2.2 of Section 2.

Theorem 3.1. *If Φ_Λ is a contractive Schur multiplier and $0 \in \Lambda$, then Λ is a semigroup with the left and right cancellation property.*

Following the proof of Theorem 2.1, where Theorem 3.1 replaces Proposition 2.2, we see that if Φ_Λ is contractive, then $\Lambda = \bigcup_k \gamma_k \Lambda_k$, where the Λ_k's are semigroups with the left and right cancellation property, and the γ_k's are orthogonal words with no final segment in Λ_k. We use the argument of Example 2.8, which only assumes that Φ_Λ is contractive, to conclude that all the Λ_k's are equal to each other. Hence, we get,

Theorem 3.2. *If Φ_Λ is a contractive Schur multiplier, then $\Lambda = \bigcup_k \gamma_k \Lambda_1$ for some semigroup Λ_1 with the left and right cancellation property, and some orthogonal words γ_k with no final segment in Λ_1. Consequently, Φ_Λ is contractive if and only if Φ_Λ is completely contractive.*

The proof of Theorem 3.1 requires several steps. We will present them separately, since they might be interesting on their own.

For the remaining of this Section assume that $0 \in \Lambda$ and that Φ_Λ is a contractive Schur multiplier.

Step 1. *(Powers) If $\alpha \in \Lambda$, then $\alpha^k \in \Lambda$ for every $k \geq 0$.*

Proof. Define $J_\alpha : A \to A_n$ and $Q_\alpha : A_n \to A$ by

$$J_\alpha z^k = e_\alpha^{\otimes k}, \quad k \geq 0, \qquad \text{and} \qquad Q_\alpha(e_\beta) = \begin{cases} z^k, & \text{if } \beta = \alpha^k, \\ 0, & \text{otherwise.} \end{cases}$$

Let $\Psi = Q_\alpha \Phi_\Lambda J_\alpha$. Since J_α and Q_α are contractions, Ψ is a contractive multipier of A. Notice that Ψ is an indicator multiplier indexed by $\sigma = \{k \in \mathbb{N} : \alpha^k \in \Lambda\}$.

Since A is commutative, Ψ is also completely contractive. Since $0 \in \sigma$, it follows from Proposition 2.2 that σ is a semigroup of $\mathbb{N}_0 = \{0, 1, 2, \cdots\}$ with the left and right cancellation property. Since $1 \in \sigma$, then $\sigma = \mathbb{N}_0$. Therefore, $\alpha^k \in \Lambda$ for every $k \geq 0$. □

Step 2. *(More powers) If $\alpha, \alpha\beta \in \Lambda$, then $\alpha\beta^k \in \Lambda$ for every $k \geq 0$.*

Proof. Let $\Psi = S_\alpha^* \Phi_\Lambda S_\alpha$ and notice that $\Psi(e_\gamma) = e_\gamma$ if $\alpha\gamma \in \Lambda$, and $\Psi(e_\gamma) = 0$ if $\alpha\gamma \notin \Lambda$. Since S_α and S_α^* are contractions, Ψ is a contractive multiplier indexed by the subset $\tilde{\Lambda} = \{\gamma : \alpha\gamma \in \Lambda\} \subset \mathbb{F}_n^+$. By assumption, $0, \beta \in \tilde{\Lambda}$. Hence, it follows from Step 1 applied to $\tilde{\Lambda}$ that for every $k \geq 0$, $\beta^k \in \tilde{\Lambda}$ and therefore $\alpha\beta^k \in \Lambda$. □

For Steps 3, 4, 5, and 7 let $p(z) = \frac{1-z}{2}$ and $q(z) = \frac{1}{N}\sum_{k=1}^N z^k$. Then $\|p\|_A = \|q\|_A = 1$ and $\|q\|_2 = \frac{1}{\sqrt{N}}$. Notice that q "peaks" around $z = 1$ and $p(1) = 0$. One can check that for every $\epsilon > 0$, there exists N such that $\|p \pm q\|_A \leq 1 + \epsilon$. If $\beta \in \mathbb{F}_n^+$, then $\|p(e_\beta) \pm q(e_\beta)\|_{A_n} = \|p \pm q\|_A \leq 1 + \epsilon$. Moreover, following the argument of Example 2.8, we see that if γ_1, γ_2 are orthogonal words, then $\|e_{\gamma_1} \otimes p(e_\beta) + e_{\gamma_2} \otimes q(e_\beta)\| \leq 1 + \epsilon$ and $\|e_{\gamma_1} + e_{\gamma_2} \otimes q(e_\beta)\| = \sqrt{\frac{5}{4}}$. For Steps 3,4, and 7 it suffices to take $1 + \epsilon < \sqrt{\frac{5}{4}}$. For Step 5 we need a more precise estimate, which is derived from the following elementary Lemma

Lemma 3.3. *Let $p(z), q(z) \in A$ be such that $\|p\|_A = \|q\|_A = 1$ and $\|p \pm q\|_A \leq 1 + \epsilon$. Let r be a unit vector in $\ell_2(\mathbb{F}_n^+)$. If $\|q(e_\beta) \otimes r\|_2 \geq 1 - \epsilon$, then $\|p(e_\beta) \otimes r\|_2 \leq 2\sqrt{\epsilon}$.*

Proof. Since $\|p(e_\beta) \pm q(e_\beta)\|_{A_n} = \|p \pm q\|_A \leq 1 + \epsilon$, then $\|p(e_\beta) \otimes r\|_2^2 + \|q(e_\beta) \otimes r\|_2^2 = \frac{1}{2}\|p(e_\beta) \otimes r + q(e_\beta) \otimes r\|_2^2 + \frac{1}{2}\|p(e_\beta) \otimes r - q(e_\beta) \otimes r\|_2^2 \leq \frac{1}{2}[\|p+q\|^2 + \|p-q\|^2] \leq (1+\epsilon)^2$. Since $\|q(e_\beta) \otimes r\|_2 \geq 1 - \epsilon$, then $\|p(e_\beta) \otimes r\|_2^2 \leq (1+\epsilon)^2 - (1 - \epsilon^2) = 4\epsilon$. □

Step 3. *(Products of orthogonal words) If α and β are orthogonal words and $\alpha, \beta \in \Lambda$, then $\alpha\beta \in \Lambda$.*

Proof. Let $\alpha, \beta \in \Lambda$ be orthogonal words and suppose, contrary to the statement of Step 3, that $\alpha\beta \notin \Lambda$. By Step 1, $\beta^k \in \Lambda$ for each $k \geq 0$. Then

$$\Phi_\Lambda\left(e_\alpha \otimes \frac{e_0 - e_\beta}{2} + e_\beta \otimes q(e_\beta)\right) = \frac{e_\alpha}{2} + e_\beta \otimes q(e_\beta).$$

Since $\|e_\alpha \otimes p(e_\beta) + e_\beta \otimes q(e_\beta)\| \leq 1 + \epsilon < \sqrt{\frac{5}{4}}$ and $\|\frac{e_\alpha}{2} + e_\beta \otimes q(e_\beta)\| = \sqrt{\frac{5}{4}}$, then $\|\Phi_\Lambda\| > 1$. This contradicts the assumption that Φ_Λ is contractive. □

Step 4. *(Right cancellation property for orthogonal words) If* α *and* β *are orthogonal words and* $\alpha, \beta\alpha \in \Lambda$*, then* $\beta \in \Lambda$*.*

Proof. Let $\alpha, \beta\alpha \in \Lambda$ and suppose, contrary to the statement of Step 4, that $\beta \notin \Lambda$. Then

$$\Phi_\Lambda\left(e_\beta \otimes \frac{1-e_\alpha}{2} + e_\alpha \otimes q(e_\alpha)\right) = \frac{e_{\beta\alpha}}{2} + e_\alpha \otimes q(e_\alpha).$$

Since $\|e_\beta \otimes p(e_\alpha) + e_\alpha \otimes q(e_\alpha)\| \leq 1+\epsilon < \sqrt{\frac{5}{4}}$ and $\|\frac{e_{\beta\alpha}}{2} + e_\alpha \otimes q(e_\alpha)\| = \sqrt{\frac{5}{4}}$, then $\|\Phi_\Lambda\| > 1$. This contradicts the assumption that Φ_Λ is contractive. □

Step 5. *(Left cancellation property for orthogonal words) If* α *and* β *are orthogonal words and* $\alpha, \alpha\beta \in \Lambda$*, then* $\beta \in \Lambda$*.*

Proof. Let $\alpha, \alpha\beta \in \Lambda$ and suppose, contrary to the statement of Step 5, that $\beta \notin \Lambda$. Let $\lambda = \frac{2}{\epsilon}$ and consider $\varphi = p(e_\beta) + \lambda e_\alpha \otimes q(e_\beta)$. It follows from Step 2 that $\alpha\beta^k \in \Lambda$ for each $k \geq 0$. Then $\Phi_\Lambda(\varphi) = \frac{e_0}{2} + \lambda e_\alpha \otimes q(e_\beta)$.

It is easy to check that φ and $\Phi_\Lambda(\varphi)$ are normed by unit vectors of the form $r = r(e_\alpha, e_\beta)$ with variables in e_α and e_β only. To see this, follow the argument of Proposition 12 of [A]. Moreover, we can assume that the constant term of r is zero. Indeed, if the constant term of r is not zero, take $r \otimes e_\alpha$ (note that $\forall \psi \in A_n, \|\psi \otimes r\|_2 = \|\psi \otimes (r \otimes e_\alpha)\|_2$). Hence, we will only consider vectors r of the form

$$r = e_\alpha \otimes r_1 + e_\beta \otimes r_2, \qquad \text{where} \qquad \|r_1\|_2^2 + \|r_2\|_2^2 = 1.$$

By Proposition 17 of [A], or by Example 6 of [AP], we have that $\|q(e_\beta) \otimes e_\alpha\|_{A_n} = \|q(e_\beta)\|_2 = \|q\|_2$. Hence, $\|e_\alpha \otimes q(e_\beta) \otimes e_\alpha \otimes r_1\|_2 \leq \|q\|_2 \|r_1\|_2$. Since we can choose $\|q\|_2$ as small as we want, we see that $e_\alpha \otimes q(e_\beta) \otimes e_\alpha \otimes r_1 \approx 0$. Therefore,

$$\varphi \otimes r \approx e_\alpha \otimes \left[\frac{r_1}{2} + \lambda q(e_\beta) \otimes e_\beta \otimes r_2\right] + e_\beta \otimes \frac{e_0 - e_\beta}{2} \otimes r_2, \qquad \text{and}$$

$$\Phi_\Lambda(\varphi) \otimes r \approx e_\alpha \otimes \left[\frac{r_1}{2} + \lambda q(e_\beta) \otimes e_\beta \otimes r_2\right] + \frac{e_\beta}{2} \otimes r_2.$$

Clearly, $\|\Phi_\Lambda(\varphi)\|_{A_n} > \lambda$. If $r = e_\alpha \otimes r_1 + e_\beta \otimes r_2$ satisfies $\|r\|_2 = 1$ and $\|\varphi \otimes r\|_2 > \lambda$, then $\lambda \|q(e_\beta) \otimes e_\beta \otimes r_2\|_2 > \lambda - 2$. Therefore, $\|q(e_\beta) \otimes e_\beta \otimes r_2\|_2 > 1 - \frac{2}{\lambda} = 1 - \epsilon$. By Lemma 3.3, we see that $\|\frac{1}{2}(e_0 - e_\beta) \otimes r_2\| \leq 2\sqrt{\epsilon}$. It is also clear that $\|r_2\|_2 \geq 1 - \epsilon$. Since $\|r_1\|_2^2 + \|r_2\|_2^2 = 1$, then $\|r_1\|_2 \leq 2\sqrt{\epsilon}$.

Let $\mu = \sup\{\|\frac{r_1}{2} + \lambda q(e_\beta) \otimes e_\beta \otimes r_2\|_2 : \|r_1\|_2^2 + \|r_2\|_2^2 = 1\}$. Then,

$$\|\varphi\|_{A_n} \leq \sqrt{\mu^2 + 4\epsilon} + 2\|q\|_2\sqrt{\epsilon}, \qquad \text{and}$$

$$\|\Phi_\Lambda(\varphi)\|_{A_n} \geq \sqrt{\mu^2 + \frac{(1-\epsilon)^2}{4}} - 2\|q\|_2\sqrt{\epsilon}.$$

If we choose $\|q\|_2$ and ϵ small enough, we get that $\|\Phi_\Lambda\| > 1$. □

Step 6. *(Left Cancellation Property) If $\alpha, \alpha\beta \in \Lambda$, then $\beta \in \Lambda$.*

Proof. If α and β are orthogonal, if follows from Step 5 that $\beta \in \Lambda$. If α and β are commutative, there exist $\gamma \in \mathbb{F}_n^+$ and $k, l \in \mathbb{N}$ such that $\alpha = \gamma^k$ and $\beta = \gamma^l$. An argument similar to the one used in Step 1 gives that $\sigma_\gamma = \{m : \gamma^m \in \Lambda\} \subset \mathbb{N}_0$ is closed under products and has the left and right cancellation property. Since we assumed that $k, k+l$ belong to this set, we get that $l \in \sigma_\gamma$. Therefore, $\beta \in \Lambda$.

Assume that α and β are non-commutative and non-orthogonal. Find $k_0, l_0 \in \mathbb{N}$ such that α^{k_0} is orthogonal to β^{l_0} (take for instance k_0, l_0 so that α^{k_0} and β^{l_0} have the same length). Clearly, $\gamma\alpha^k$ is orthogonal to $\gamma\beta^l$ for every $k \geq k_0$, $l \geq l_0$, and $\gamma \in \mathbb{F}_n^+$. In particular, $\alpha\alpha^{k_0} = \alpha^{k_0+1}$ is orthogonal to $\alpha\beta^l$ for every $l \geq l_0$. By Step 2, $\alpha\beta^l \in \Lambda$. Hence, by Step 3, $(\alpha^{k_0+1})(\alpha\beta^l) = \alpha^{k_0+2}\beta^l \in \Lambda$. Since α^{k_0+2} is orthogonal to β^l, we use Step 5 to get $\beta^l \in \Lambda$. Hence, $\beta^l \in \Lambda$ for every $l \geq l_0$. We follow the argument used in Step 1 to conclude that $\sigma_\beta = \{l : \beta^l \in \Lambda\} \subset \mathbb{N}_0$ is closed under products and has the left and right cancellation property. Since $l_0, l_0 + 1$ belong to this set, we see that $1 \in \sigma_\beta$ also. Therefore, $\beta \in \Lambda$. ☐

Step 7. *(Right Cancellation Property) If $\alpha, \beta\alpha \in \Lambda$, then $\beta \in \Lambda$.*

Proof. We assume, as we did in Step 6, that α and β are non-commutative and non-orthogonal. Find $k, l \in \mathbb{N}$ such that $(\beta\alpha)^k$ is orthogonal to α^l. Consider $e_{(\beta\alpha)^k\beta} \otimes \frac{e_0 - e_\alpha}{2} + e_{\alpha^l} \otimes q(e_\alpha)$. If $(\beta\alpha)^k\beta \notin \Lambda$, we would have that $\|\Phi_\Lambda(e_{(\beta\alpha)^k\beta} \otimes \frac{e_0 - e_\alpha}{2} + e_{\alpha^l} \otimes q(e_\alpha))\| = \|\frac{1}{2}e_{(\beta\alpha)^k\beta} + e_{\alpha^l} \otimes q(e_\alpha)\| = \sqrt{\frac{5}{4}}$, which would imply that $\|\Phi_\Lambda\| > 1$. Therefore, $(\beta\alpha)^k\beta \in \Lambda$. Since $(\beta\alpha)^k \in \Lambda$, we conclude from Step 6 that $\beta \in \Lambda$. ☐

Step 8. *(Products) If $\alpha, \beta \in \Lambda$, then $\alpha\beta \in \Lambda$.*

Proof. We assume again that α and β are non-commutative and non-orthogonal. Hence, $\beta = \alpha\gamma$ for some γ, which belongs to Λ by Step 6.

Since we assumed that the length of β is greater that the length of α, we must consider $\alpha\beta$ and $\beta\alpha$. We will show first that $\beta\alpha = \alpha\gamma\alpha \in \Lambda$. Since $\beta = \alpha\gamma \in \Lambda$, then $\beta^2 = \alpha\gamma\alpha\gamma \in \Lambda$. Since $\gamma \in \Lambda$, we get from Step 7 that $\alpha\gamma\alpha \in \Lambda$. We will now show that $\alpha\beta = \alpha\alpha\gamma \in \Lambda$. Find $k, l \in \mathbb{N}$ such that α^k is orthogonal to γ^l. Then, by Step 3, $\alpha^k\gamma^l \in \Lambda$. Since $\gamma^{l-1} \in \Lambda$, then, by Step 7, $\alpha^k\gamma \in \Lambda$. Since $\alpha^{k-2} \in \Lambda$, then, by Step 6, $\alpha^2\gamma \in \Lambda$. ☐

4. Multipliers Associated with Representations

In this section we show that coefficients of unitary representations of the free semigroup $\mathbb{F}_n^+$ induce completely bounded multipliers of A_n. The proof is elementary.

Proposition 4.1. *Let $\sigma : \mathbb{F}_n^+ \to B(H)$ be a unitary representation and let $\xi, \eta \in H$. Then the map Φ defined by $\Phi(\lambda(\alpha)) = \langle\sigma(\alpha)\xi, \eta\rangle\lambda(\alpha)$ is a completely bounded multiplier of A_n satisfying $\|\Phi\|_{cb} \leq \|\xi\|_2\|\eta\|_2$.*

Proof. We will show that Φ is the restriction of a Schur multiplier M on $B(\ell_2(\mathbb{F}_n^2))$ with norm $\|M\| \leq \|\xi\|_2\|\eta\|_2$. Define $x, y : \mathbb{F}_n^+ \to H$ by $x(\beta) = (\sigma(\beta))^*\xi$ and $y(\beta) =$

$(\sigma(\beta))^*\eta$, and define M on $B(\ell_2(\mathbb{F}_n^+))$ by $\langle MT\delta_\beta, \delta_\gamma\rangle = \langle x(\beta), x(\eta)\rangle\langle T\delta_\beta, \delta_\gamma\rangle$. By Theorem 1.2, M is a Schur multiplier with norm $\|M\|_{cb} \leq \|\xi\|_2\|\eta\|_2$. Since

$$\begin{aligned}\langle M\lambda(\alpha)\delta_\beta, \delta_{\alpha\beta}\rangle &= \langle x(\beta), y(\alpha\beta)\rangle = \langle(\sigma(\beta))^*\xi, (\sigma(\alpha\beta))^*\eta\rangle \\ &= \langle\sigma(\alpha\beta)(\sigma(\beta))^*\xi, \eta\rangle = \langle\sigma(\alpha)\xi, \eta\rangle,\end{aligned}$$

we conclude that $M\lambda(\alpha) = \langle\sigma(\alpha)\xi, \eta\rangle\lambda(\alpha)$. □

Recall that $B(\mathbb{F}_n^+)$ is the space of coefficients of contractive representations of $\mathbb{F}_n^+$. That is, $u : \mathbb{F}_n^+ \to \mathbb{C}$ belongs to $B(\mathbb{F}_n^+)$ if and only if there exist a Hilbert space H, a contractive representation $\pi : \mathbb{F}_n^+ \to B(H)$, and two vectors $\xi, \eta \in \mathbb{F}_n^+$ such that

$$\forall\alpha \in \mathbb{F}_n^+, \quad u(\alpha) = \langle\pi(\alpha)\xi, \eta\rangle. \tag{3}$$

The norm of u is given by

$$\|u\|_{B(\mathbb{F}_n^+)} = \inf\{\|\xi\|\|\eta\| : (3) \text{ holds }\}.$$

One can check that the coefficients of contractive representations of $\mathbb{F}_n^+$ coincide with the coefficients of unitary representations of $\mathbb{F}_n^+$; and hence, Proposition 4.1 is also true for contractive representations. To see this, suppose that $u(\alpha) = \langle\pi(\alpha)\xi, \eta\rangle$, where $\pi : \mathbb{F}_n^+ \to B(H)$ is a contractive representation and $\xi, \eta \in H$. The unital representation π is determined by the contractions $\pi(g_i) = T_i \in B(H)$. Assume that H is infinitely dimensional and use the Dilation Theorem to find unitary operators $U_i \in B(H \oplus H \oplus H)$ such that the U_i's are upper triangular and the (2,2)-component of U_i is T_i. Define the unitary representation $\sigma : \mathbb{F}_n^+ \to B(H \oplus H \oplus H)$ by $\pi(g_i) = U_i$, and let $\bar{\xi} = (0, \xi, 0), \bar{\eta} = (0, \eta, 0)$ be vectors in $H \oplus H \oplus H$. Then,

$$\forall\alpha \in \mathbb{F}_n^+, \quad \langle\pi(\alpha)\xi, \eta\rangle = \langle\sigma(\alpha)\bar{\xi}, \bar{\eta}\rangle.$$

It is not difficult to see that $B(\mathbb{F}_n^+)$ is a Banach algebra and that it is the Banach space dual of the algebra generated by the universal isometric representation of $\mathbb{F}_n^+$. This is analogous to the situation for discrete groups, where $B(G)$ is the dual of the full C^*-algebra of G, $C^*(G)$.

Recall also that $u : \mathbb{F}_n^+ \to \mathbb{C}$ is a bounded Schur multiplier of A_n if the operator Φ_u on A_n defined by $\Phi_u(\lambda(\alpha)) = u(\alpha)\lambda(\alpha)$ is bounded. $M(\mathbb{F}_n^+)$ is the space of bounded Schur multipliers with norm $\|u\|_{M(\mathbb{F}_n^+)} = \|\Phi_u\|$. And $M_0(\mathbb{F}_n^+)$ is the space of those u's such that Φ_u is completely bounded. The norm on $M_0(\mathbb{F}_n^+)$ is given by $\|u\|_{M_0(\mathbb{F}_n^+)} = \|\Phi_u\|_{cb}$. Proposition 4.1 states that $B(\mathbb{F}_n^+) \subset M_0(\mathbb{F}_n^+)$ and that the inclusion is contractive. We will show that $M_0(\mathbb{F}_n^+) \setminus B(\mathbb{F}_n^+) \neq \emptyset$.

Example 4.2. *Let $n = 2$ and $\Lambda = g_2\mathbb{F}_2^+$. Then Φ_Λ is completely bounded but Φ_Λ is not a coefficient of an isometric representation of $\mathbb{F}_2^+$.*

Proof. It is easy to see that $\|\Phi_\Lambda\|_{cb} \leq 2$. Suppose that there exists an isometric representation $\pi : \mathbb{F}_2^+ \to B(H)$ and $\xi, \eta \in H$ such that

$$\langle\pi(\alpha)\xi, \eta\rangle = \begin{cases} 1, & \text{if } \alpha \in \Lambda, \\ 0, & \text{otherwise.}\end{cases}$$

Clearly, π is determined by the isometries $T_1 = \pi(g_1)$ and $T_2 = \pi(g_2)$. Hence, for any $\alpha \in \mathbb{F}_2^+$, $\langle T_2 T_\alpha \xi, \eta\rangle = 1$ and $\langle T_1 T_\alpha \xi, \eta\rangle = 0$.

Let $\mathcal{C} = \overline{\text{conv}}\{T_2 T_\alpha \xi : \alpha \in \mathbb{F}_2^+\}$ and find $z \in \mathcal{C}$ satisfying

$$\|z\| = \inf\{\|w\| : w \in \mathcal{C}\}. \tag{4}$$

It is easy to see that such a z exists and is unique. Notice that for any $\alpha \in \mathbb{F}_2^+$, $T_2 T_\alpha z \in \mathcal{C}$, $\langle T_2 T_\alpha z, \eta\rangle = 1$, and $\langle T_1 T_\alpha z, \eta\rangle = 0$. Since $T_2 T_\alpha$ is an isometry, then $\|T_2 T_\alpha z\|_2 = \|z\|_2$. Moreover, since the minimum of (4) is unique, we conclude that $\forall \alpha \in \mathbb{F}_2^+, T_2 T_\alpha z = z$. In particular, $T_2 z = T_2 T_1 z = z$. Hence, $T_2(z - T_1 z) = 0$. Since T_2 is one-to-one, we get that $T_1 z = z$. But this implies that $0 = \langle T_1 z, \eta\rangle = \langle z, \eta\rangle = 1$. Therefore, Φ_Λ is not the coefficient of a contractive representation. □

The space of coefficients of the left regular representation plays an interesting role. We say that $u \in B^{(\lambda)}(\mathbb{F}_n^+)$ if there exist $\varphi, \psi \in \ell_2(\mathbb{F}_n^+)$ such that

$$\forall \alpha \in \Lambda, \quad u(\alpha) = \langle \lambda(\alpha)\varphi, \psi\rangle = \langle e_\alpha \otimes \varphi, \psi\rangle, \tag{5}$$

and

$$\|u\|_{B^{(\lambda)}(\mathbb{F}_n^+)} = \inf\{\|\varphi\|_2 \|\psi\|_2 : (5) \text{ holds }\}.$$

Recently, Davidson and Pitts [DP1] proved that $B^{(\lambda)}(\mathbb{F}_n^+)$ is the predual of $\mathcal{F}^\infty(H_n)$, the wot-closure of A_n. Davidson and Pitts called $\mathcal{F}^\infty(H_n)$ "free semigroup algebras" and denoted them by $\mathcal{L}_n$. If G is a countable group, $B^{(\lambda)}(G)$ is called the Fourier algebra of G and is denoted by $A(G)$. It is known that $A(G)$ is the predual of $VN(G)$, (the von Neumann algebra generated by the left regular representation of G), and that $A(G)$ is a closed ideal in $B(G)$, (see [E]).

We will show that $B^{(\lambda)}(\mathbb{F}_n^+)$ is an ideal in $B(\mathbb{F}_n^+)$. However, unlike the situation for discrete groups, the norms do not coincide. Indeed, if

$$u : \mathbb{F}_n^+ \to \mathbb{C} \quad \text{is defined by} \quad u(\alpha) = \begin{cases} 1, & \text{if } |\alpha| = 1, \\ 0, & \text{otherwise }, \end{cases}$$

one can check that $\|u\|_{B(\mathbb{F}_n^+)} = 1$ and $\|u\|_{B^{(\lambda)}(\mathbb{F}_n^+)} = \sqrt{n}$.

We need Fell's absorption principle. If $\pi : \mathbb{F}_n^+ \to B(H)$ is a unitary representation, then $\lambda \otimes \pi$ is unitarily equivalent to $\lambda \otimes I$. That is, $\lambda \otimes \pi(\alpha) = V^*(\lambda \otimes I)(\alpha)V$ for some unitary operator on $\ell_2(\mathbb{F}_n^+) \otimes_2 H$ (see [Fe]).

Proposition 4.3. *$B^{(\lambda)}(\mathbb{F}_n^+)$ is an ideal in $B(\mathbb{F}_n^+)$.*

Proof. Let $u \in B^{(\lambda)}(\mathbb{F}_n^+)$ and $v \in B(\mathbb{F}_n^+)$. Find a Hilbert space with orthonormal basis $\{\delta_k : k \geq 1\}$, a unitary representation $\pi : \mathbb{F}_n^+ \to B(H)$, and two vectors $\xi, \eta \in H$ such that for each $\alpha \in \mathbb{F}_n^+$, $v(\alpha) = \langle \pi(\alpha)\xi, \eta\rangle$. Find also $\varphi, \psi \in \ell_2(\mathbb{F}_n^+)$ such that $u(\alpha) = \langle \lambda(\alpha)\varphi, \psi\rangle$.

Use Fell's absorption principle to find a unitary map V on $\ell_2(\mathbb{F}_n^+) \otimes_2 H$ such that $V^*(\lambda \otimes \pi)V = \lambda \otimes I$. Let $\alpha \in \mathbb{F}_n^+$. Then,

$$\begin{aligned} u(\alpha)v(\alpha) &= \langle \lambda(\alpha)\varphi, \psi\rangle\langle \pi(\alpha)\xi, \eta\rangle \\ &= \langle (\lambda \otimes \pi)(\alpha)\varphi \otimes \xi, \psi \otimes \eta\rangle \\ &= \langle (\lambda \otimes I)(\alpha)V(\varphi \otimes \xi), V(\psi \otimes \eta)\rangle. \end{aligned}$$

Write $V(\varphi\otimes\xi)=\sum_k\varphi_k\otimes\delta_k$ and $V(\psi\otimes\eta)=\sum_k\psi_k\otimes\delta_k$ for some $\varphi_k,\psi_k\in\ell_2(\mathbb{F}_n^+)$. We employ the technique used by Davidson and Pitts to show that the dual of $B^{(\lambda)}(\mathbb{F}_n^+)$ is $\mathcal{L}_n$. This one consists in replacing the δ_k's with orthogonal elements from $\ell_2(\mathbb{F}_n^+)$. For each k, let $\theta_k=e_2\otimes e_1^k$. Notice that for every $\phi_1,\phi_2\in\ell_2(\mathbb{F}_n^+)$, $\langle\phi_1,\phi_2\rangle=\langle\phi_1\otimes\theta_k,\phi_2\otimes\theta_k\rangle$. Moreover, if $k\neq j$, then $\langle\phi_1\otimes\theta_k,\phi_2\otimes\theta_j\rangle=0$. Then, it is easy to see that $\bar\varphi=\sum_k\varphi_k\otimes\theta_k$ and $\bar\psi=\sum_k\psi_k\otimes\theta_k$ satisfy $\|\bar\varphi\|_2=\sqrt{\sum_k\|\varphi_k\|_2^2}=\|\sum_k\varphi_k\otimes\delta_k\|=\|V(\varphi\otimes\xi)\|=\|\varphi\|_2\|\xi\|_2$ and $\|\bar\psi\|=\|\psi\|_2\|\eta\|_2$. Moreover,

$$\begin{aligned}
u(\alpha)v(\alpha)&=\langle(\lambda(\alpha)\otimes I)V(\varphi\otimes\xi),V(\psi\otimes\eta)\rangle\\
&=\sum_{k=1}^{\infty}\langle\lambda(\alpha)\varphi_k,\psi_k\rangle\\
&=\Big\langle\lambda(\alpha)\Big(\sum_k\varphi_k\otimes\theta_k\Big),\Big(\sum_j\psi_j\otimes\theta_j\Big)\Big\rangle\\
&=\langle\lambda(\alpha)\bar\varphi,\bar\psi\rangle.
\end{aligned}$$

Since $\bar\varphi,\bar\psi\in\ell_2(\mathbb{F}_n^+)$, then $uv\in B^{(\lambda)}(\mathbb{F}_n^+)$ and $\|uv\|_{B^{(\lambda)}(\mathbb{F}_n^+)}\leq\|u\|_{B^{(\lambda)}(\mathbb{F}_n^+)}\|v\|_{B(\mathbb{F}_n^+)}$.
□

5. Representations of A_n

Problem 6.3 of [Po7] asked if every contractive representation of A_n, $n\geq 2$, was also completely contractive. In this section we answer this question in the negative.

Proposition 5.1. *There exists a contractive representation of A_n that is not completely contractive.*

We will also show that there are very simple bounded representations of A_n, $n\geq 2$, that are not completely bounded. This result is not new. The examples used recently by Pisier [P2] to answer Halmos' Similarity Problem work also in A_n, $n\geq 2$. However, they are more complicated.

Recall that the unitary flip Θ is defined by $\Theta(e_{i_1}\otimes\cdots e_{i_n})=e_{i_n}\otimes\cdots\otimes e_{i_1}$ and is denoted by $\Theta(\varphi)=\tilde\varphi$.

Example 5.2. *Let $n\geq 2$ and $\frac{1}{\sqrt{n}}<\delta<1$. Let $\pi:A_n\to B(\ell_n(\mathbb{F}_n^+))$ be the unital representation induced by $\pi(S_i)=\delta S_i^*$, $i\leq n$, where $S_i=\lambda(g_i)$. Then π is a bounded representation but it is not completely bounded.*

Proof. Let $p\in\mathcal{P}$, $\|p\|_{A_n}=1$. Write $p=p_0+p_1+p_2+\cdots+p_N$, where $p_k\in\operatorname{span}\{e_\alpha:|\alpha|=k\}$. From Proposition 16 of [A], we have that $\|p_k\|_{A_n}=\|p_k\|_2$. Notice that

$$\pi(p)=p_0+\delta p_1(S_1^*,\cdots,S_n^*)+\delta^2p_2(S_1^*,\cdots,S_n^*)+\cdots+\delta^Np_N(S_1^*,\cdots,S_n^*).$$

Since $(S_{i_1}^*S_{i_2}^*\cdots S_{i_k}^*)^*=S_{i_k}\cdots S_{i_2}S_{i_1}$, we see that

$$\|p_k(S_1^*,\cdots,S_n^*)\|_{B(\ell_2(\mathbb{F}_n^+))}=\|(p_k(S_1^*,\cdots,S_n^*))^*\|_{B(\ell_2(\mathbb{F}_n^+))}=\|\tilde p_k(S_1,\cdots,S_n)\|_{A_n},$$

where $\tilde{p}_k$ is the "flip" of p_k. Using Proposition 16 of [A] again, we get that $\|\tilde{p}_k(S_1, \cdots, S_n)\|_{A_n} = \|\tilde{p}_k\|_2 = \|p_k\|_2 \leq 1$. Therefore,

$$\|\pi(p)\| \leq \sum_{k=0}^{N} \delta_k \|p_k(S_1^*, \cdots, S_n^*)\| \leq \sum_{k=0}^{n} \delta^k < \frac{1}{1-\delta},$$

and π is a bounded representation.

We will now check that π is not completely bounded. Fix k and let $\alpha_1, \alpha_2, \cdots, \alpha_{n^k}$ be an ennumeration of the set $\{\alpha : |\alpha| = k\}$. Then

$$\|[S_{\alpha_1} S_{\alpha_2} \cdots S_{\alpha_{n^k}}]\| = \left\| \sum_{i=1}^{n^k} S_{\alpha_i} S_{\alpha_i}^* \right\|^{\frac{1}{2}} = 1, \text{ and}$$

$$\|[S_{\alpha_1}^* S_{\alpha_2}^* \cdots S_{\alpha_{n^k}}^*]\| = \left\| \sum_{i=1}^{n^k} S_{\alpha_i}^* S_{\alpha_i} \right\|^{\frac{1}{2}} = \sqrt{n^k}.$$

Since $(1 \otimes \pi)[S_{\alpha_1} \cdots S_{\alpha_{n^k}}] = \delta^k [S_{\tilde{\alpha}_1} \cdots S_{\tilde{\alpha}_{n^k}}]$, we conclude that $\|\pi\|_{cb} \geq (\delta\sqrt{n})^k$. $\square$

To complete the proof of Proposition 5.1, we need to study the structure of some natural subspaces of A_n. Let $\varphi \in A_n$, and define

$$\ell_2(\mathbb{F}_\varphi) = \overline{\text{span}}\{e_0, \varphi, \varphi^{\otimes 2}, \cdots\} \subset \ell_2(\mathbb{F}_n^+),$$
$$A_\varphi = \overline{\text{span}}\{e_0, \varphi, \varphi^{\otimes 2} \cdots\} \subset A_n.$$

Let P_φ be the orthogonal projection onto $\ell_2(\mathbb{F}_\varphi)$. Recall that $\varphi \in A_n$ is inner iff for every $\psi_1, \psi_2 \in \ell_2(\mathbb{F}_n^+)$, $\langle \varphi \otimes \psi_1, \varphi \otimes \psi_2 \rangle = \langle \psi_1, \psi_2 \rangle$; and that $\tilde{\varphi} \in A_n$ is inner iff for every $\psi_1, \psi_2 \in \ell_2(\mathbb{F}_n^+)$, $\langle \psi_1 \otimes \varphi, \psi_2 \otimes \varphi \rangle = \langle \psi_1, \psi_2 \rangle$

Proposition 5.3. *Let φ be inner. Then the map $V_\varphi : A \to A_\varphi$, defined by $V_\varphi z^k = \varphi^{\otimes k}$, is an isometry onto. Moreover, if $\tilde{\varphi}$ is also inner, then $P_\varphi : A_n \to A_\varphi$ is contractive.*

Proof. Suppose that φ is inner, and let $p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_N z^N$ be a polynomial.

Since $\|\varphi\| = 1$, it follows from von Neumann's inequality that $\|\sum_k a_k \varphi^{\otimes k}\| \leq \|\sum_k a_k z^k\|$.

Notice that $\ell_2(\mathbb{F}_\varphi) \ominus \left[\varphi \otimes \ell_2(\mathbb{F}_\varphi)\right] = \text{span}\ \{\psi\}$ for some $\psi \in \ell_2(\mathbb{F}_\varphi)$, $\|\psi\|_2 = 1$. Since φ is inner (i.e., an isometry), then $\{\varphi^{\otimes k} \otimes \psi\}_{k=0}^{\infty}$ is an orthonormal basis of $\ell_2(\mathbb{F}_\varphi)$. Let $\epsilon > 0$, and find $q = \sum_j b_j z^j$ satisfying $\|q\|_2 = 1$ and $\|p\|_{A(z)} \leq \|pq\|_2 + \epsilon$. Since $\|\sum_j b_j [\varphi^{\otimes k} \otimes \psi]\|_2 = 1$, we see that

$$\begin{aligned}
\left\| \sum_k a_k \varphi^{\otimes k} \right\| &\geq \left\| \left(\sum_k a_k \varphi^{\otimes k} \right) \otimes \left(\sum_j b_j [\varphi^{\otimes k} \otimes \psi] \right) \right\|_2 \\
&= \left\| \sum_k \sum_j a_k b_j [\varphi^{\otimes (k+j)} \otimes \psi] \right\|_2 \\
&= \left\| \sum_k \sum_j a_k b_j z^{k+j} \right\|_2 = \|pq\|_2 \geq \|p\|_2 - \epsilon.
\end{aligned}$$

Therefore, $\|\sum_k a_k \varphi^{\otimes k}\|_{A_n} \geq \|\sum_k a_k z^k\|_{A(z)}$.

Suppose that $\tilde{\varphi}$ is inner too. Let $p \in \mathcal{P}$, and write $p = p_1 + p_2$, where $p_1 \in \ell_2(\mathbb{F}_\varphi)$ and $p_2 \in (\ell_2(\mathbb{F}_\varphi))^\perp$. From the first part of this Proposition, we see that $\|p_1\| = \sup\{\|p_1 \otimes \eta\|_2 : \|\eta\|_2 = 1, \eta \in \ell_2(\mathbb{F}_\varphi)\}$.

Let $\eta \in \ell_2(\mathbb{F}_\varphi)$. Clearly, $p_1 \otimes \eta \in \ell_2(\mathbb{F}_\varphi)$, and we claim that $p_2 \otimes \eta \in (\ell_2(\mathbb{F}_\varphi))^\perp$. Indeed, it is enough to verify that for every $k, m \geq 0$, $\langle \varphi^{\otimes k}, p_2 \otimes \varphi^{\otimes m} \rangle = 0$. Since $\tilde{\varphi}$ is inner, $p_2 \in (\ell_2(\mathbb{F}_\varphi))^\perp$, and $\langle e_0, p_2 \rangle = 0$, we get that

$$\langle \varphi^{\otimes k}, p_2 \otimes \varphi^{\otimes m} \rangle = \begin{cases} \langle \varphi^{\otimes (k-m)}, p_2 \rangle = 0, & \text{if } k \geq m, \\ \langle e_0, p_2 \otimes \varphi^{\otimes (m-k)} \rangle = 0, & \text{if } k < m. \end{cases}$$

Hence, $\|p \otimes \eta\|_2 = \sqrt{\|p_1 \otimes \eta\|_2^2 + \|p_2 \otimes \eta\|_2^2} \geq \|p_1 \otimes \eta\|_2$. Therefore, $\|p_1\| \leq \|p\|$. $\square$

Proof of Proposition 5.1. Let $T_1, T_2, \cdots T_n \in B(H)$ be operators on H satisfying

$$T_iT_j = 0, \quad \forall i, j \leq n, \quad \text{and} \quad \left\| \sum_{i=1}^n a_iT_i \right\| \leq \sqrt{\sum_{i=1}^n |a_i|^2}, \quad \forall a_i \in \mathbb{C}, i \leq n. \tag{6}$$

Let $\pi : A_n \to B(H)$ be the unital representation determined by $\pi(S_i) = T_1$. We will check that π is contractive.

Let $p \in \mathcal{P}$, $p = a_0e_0 + (a_1e_1 + \cdots + a_ne_n) +$ (higher terms). Notice that

$$\pi(p) = p(T_1, \cdots, T_n) = a_0I + (a_1T_1 + \cdots a_nT_n) = a_0I + \sqrt{\sum_{i=1}^n |a_i|^2}\, T,$$

where $T = (a_1T_1 + a_2T_2 + \cdots a_nT_n)/\sqrt{\sum_{i=1}^n |a_i|^2}$. By hypothesis, $\|T\| \leq 1$.

Let $\varphi = (a_1e_1 + \cdots a_ne_n)/\sqrt{\sum_{i=1}^n |a_i|^2}$. Notice that φ and $\tilde{\varphi}$ are inner and that

$$P_\varphi(p) = a_0e_0 + \sqrt{\sum_{i=1}^n |a_i|^2}\varphi + b_2\varphi^{\otimes 2} + \cdots + b_N\varphi^{\otimes N},$$

for some $b_2, b_3, \cdots b_N \in \mathbb{C}$. Then,

$$\begin{aligned} \|\pi(p)\| &= \left\| a_0I + \sqrt{\sum |a_i|^2}\, T \right\| \\ &= \left\| a_0I + \sqrt{\sum |a_i|^2}\, T + b_2T^2 + \cdots + b_NT^N \right\|, \qquad \text{since } T^2 = \cdots = T^N = 0, \\ &\leq \left\| a_0 + \sqrt{\sum |a_i|^2}\, z + b_2z^2 + \cdots + b_Nz^N \right\|, \qquad \text{by von Neumann's ineq.,} \\ &= \Bigg\| a_0e_0 + \sqrt{\sum |a_i|^2}\, \varphi + b_2\varphi^{\otimes 2} + \cdots + b_N\varphi^{\otimes N} \|, \qquad \text{by Lemma 5.3,} \\ &= \|P_\varphi(p)\| \leq \|p\|. \end{aligned}$$

To finish the proof, it is enough to find n operators satisfying (6) such that $\|[T_1T_2 \cdots T_n]\| > 1$. Choose T_i to be basic elements of the row Hilbert space, "shifted" to the right (to insure that $T_iT_j = 0$). That is, $T_i = e_{1\, i+1} \in M_{n+1}$, where M_{n+1} is the set of $(n+1) \times (n+1)$ matrices. Clearly, these operators satisfy (6), and hence, the unital representation $\pi : A_n \to M_{n+1}$ determined by $\pi(\lambda(g_i)) = T_i$ is contractive. However, $\|[\lambda(g_1) \cdots \lambda(g_n)]\| = 1$ and $\|[T_1 \cdots T_n]\| = \sqrt{n}$. Since $(1 \otimes \pi)[\lambda(g_1) \cdots \lambda(g_n)] = [T_1 \cdots T_n]$, we get that $\|\pi\|_{cb} \geq \sqrt{n}$. $\square$

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