NONCOMMUTATIVE INTERPOLATION AND POISSON TRANSFORMS II

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Abstract. We associate to certain weights $\omega_\alpha > 0$ some weighted left creation operators $W_1, \dots, W_n$ on the full Fock space. The weighted noncommutative analytic Toeplitz algebra $F^\infty(\omega_\alpha)$ is the WOT-closure of the algebra generated by $W_1, \dots, W_n$ and the identity. Noncommutative Poisson transforms on $F^\infty(\omega_\alpha)$ are used to provide a WOT-continuous, $F^\infty(\omega_\alpha)$-functional calculus for sequences of operators satisfying certain positivity conditions.

This leads to completely isometric representations of the quotient algebra $F^\infty(\omega_\alpha)/J$ on Hilbert spaces, where J is any w^*-closed, 2-sided ideal in $F^\infty(\omega_\alpha)$. We obtain noncommutative interpolation problems of Carathéodory and of Nevanlinna-Pick in $F^\infty(\omega_\alpha)$ and in the space of multipliers of some weighted Hardy spaces in the unit ball of $\mathbb{C}^n$.

In this paper we use a technique from [APo2] to obtain simple and explicit completely isometric representations of quotients of some weighted Fock spaces $F^\infty(\omega_\alpha)$ (see Section 1 for notation) and of the space of multipliers $M(\omega_k)$ of some weighted Hardy spaces on the unit ball of $\mathbb{C}^n$ (see Section 3 for notation). The selection of the weights is motivated by a paper of Quiggin [Q].

We use these representations to obtain noncommutative Carathéodory's and Nevanlinna-Pick's interpolation theorems on $F^\infty(\omega_\alpha)$, and to obtain multivariate Carathéodory's interpolation theorems on $M(\omega_k)$. Furthermore, we give new proofs of Nevanlinna-Pick's interpolation theorems on $M(\omega_k)$ (see [Q]), and of Carathéodory's interpolation theorems on $M(\omega_k)$ when $n = 1$ (see [Mc1]). In particular, we obtain new proofs of Nevanlinna-Pick's and Carathéodory's interpolation theorem on the Dirichlet algebra. This was proved by Agler in the 80's but remains unpublished (see [Q] and [Mc1]).

The setting of this paper is the following: A is a unital w^*-closed subalgebra of $B(H)$, J is a w^*-closed 2-sided ideal of A, $\mathcal{N}_J \subset H$ is the orthogonal complement of the image of J, and $P_{\mathcal{N}_J}$ is the orthogonal projection onto $\mathcal{N}_J$. It is well known

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that the map $\Psi_J : A/J \to B(\mathcal{N}_J)$ defined by $\Psi_J(a+J) = P_{\mathcal{N}_J} a|_{\mathcal{N}_J}$ is a completely contractive representation. In [S], Sarason proved that if A is the analytic Toeplitz algebra H^∞ and J is a w^*-closed ideal of H^∞, then $\Psi_J : H^\infty/J \to B(\mathcal{N}_J)$ is a completely isometric representation. In [APo2] and [DP3], this result was extended to the noncommutative analytic Toeplitz algebra F^∞, which was introduced in [Po4] in connection with a noncommutative version of von Neumann's inequality [vN]. We gave two proofs of the representation result of the quotient algebra F^∞/J in [APo2]: one based on the noncommutative commutant lifting theorem of [Po2], and the other based on representations of quotient algebras and the noncommutative Poisson transforms of [Po7]. Davidson and Pitts [DP3] gave a different proof, which was based on the structure of invariant subspaces of F^∞ (see [Po1]), and results on ideals and $\mathbb{A}_1$-properties of F^∞ (see [DP1]).

In [APo2], we used ideas of Cole (see [Pi]) to show that if J is a w^*-closed 2-sided ideal of the unital w^*-closed subalgebra A of $B(H)$, then there exists a subspace $\mathcal{E} \subset \ell_2 \otimes H$ such that the map $\widehat{\Psi} : A/J \to B(\ell_2 \otimes H)$ defined by $\widehat{\Psi}(a+J) = P_{\mathcal{E}}(I_{\ell_2} \otimes a)|_{\mathcal{E}}$ is a completely isometric representation. We then used the noncommutative Poisson transform of [Po7] to show that if $A = F^\infty$, then there exists a w^*-continuous completely contractive map $\Phi_J : B(\mathcal{N}_J) \to B(\ell_2 \otimes H)$ so that the following diagram commutes

$$\begin{tikzcd}
A/J \arrow[rr, "\widehat{\Psi}"] \arrow[dr, "\Psi_J"'] & & B(\ell_2 \otimes_2 H) \\
& B(\mathcal{N}_J) \arrow[ur, "\Phi_J"'] &
\end{tikzcd}.$$

Since $\widehat{\Psi}$ is completely isometric and Ψ_J and Φ_J are completely contractive, we concluded that $\Psi_J : F^\infty/J \to B(\mathcal{N}_J)$ was completely isometric.

In this paper, we show that if $A = F^\infty(\omega_\alpha)$, where the weights satisfy the conditions of Section 1, and J is a w^*-closed, 2-sided ideal of $F^\infty(\omega_\alpha)$, then we can modify the noncommutative Poisson transform of [Po7] to obtain a completely contractive map $\Phi_J : B(\mathcal{N}_J) \to B(\ell_2 \otimes H)$ so that $\widehat{\Psi} = \Phi_J \circ \Psi_J$. A similar argument works in the commutative case. If $A = M(\omega_k)$, where the weights satisfy the conditions of Section 3, and J is a w^*-closed 2-sided ideal of $M(\omega_k)$, then one can modify the dilation of [Dr] to obtain a completely contractive map $\Phi_J : B(\mathcal{N}_J) \to B(\ell_2 \otimes H)$ so that $\widehat{\Psi} = \Phi_J \circ \Psi_J$. However, we will only present the arguments for $F^\infty(\omega_\alpha)$, since the commutative case follows easily from this. Furthermore, the computations in the noncommutative case are simpler.

Let us mention that when $\omega_\alpha = 1$ and $n \geq 2$, then $F^\infty(\omega_\alpha)$ coincides with the noncommutative analytic Toeplitz algebra F^∞ (see [Po4]). This algebra has been studied in several papers [Po1], [Po2], [Po3], [Po4], [Po5], [Po6], [APo1], [APo2], [DP1], [DP2], and [DP3]. In this setting, the Carathéodory interpolation problem was obtained in [Po6] using the noncommutative commutant lifting theorem [Po2] (see [SzF] and [FFr] for the classical case), and the Nevanlinna-Pick interpolation problem was proved in [APo2] and [DP3] independently.

1. Notation and preliminaries

Unless explicitly stated, n stands for a cardinal number between $1 \leq n \leq \aleph_0$. Let $\mathcal{H}_n$ be an n-dimensional Hilbert space with orthonormal basis $e_1, e_2, \dots, e_n$. We consider the Full Fock space [E] of $\mathcal{H}_n$

$$\mathcal{F}^2 = \mathcal{F}^2(\mathcal{H}_n) = \bigoplus_{k\geq 0} \mathcal{H}_n^{\otimes k},$$

where $\mathcal{H}_n^{\otimes 0} = \mathbb{C}1$ and $\mathcal{H}_n^{\otimes k}$ is the (Hilbert) tensor product of k copies of $\mathcal{H}_n$. We shall denote by $\mathcal{P}$ the set of all $p \in \mathcal{F}^2(\mathcal{H}_n)$ of the form

$$p = a_0 + \sum_{\substack{1\leq i_1,\dots,i_k\leq n\\ 1\leq k\leq m}} a_{i_1\cdots i_k} e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_k}, \quad m \in \mathbb{N},$$

where $a_0, a_{i_1\cdots i_k} \in \mathbb{C}$. The set $\mathcal{P}$ may be viewed as the algebra of the polynomials in n noncommuting indeterminates, with $p \otimes q$, $p, q \in \mathcal{P}$, as multiplication. For any bounded operators $T_1, \dots, T_n$ on a Hilbert space $\mathcal{H}$, define

$$p(T_1, \dots, T_n) := a_0 I_{\mathcal{H}} + \sum a_{i_1\cdots i_k} T_{i_1} T_{i_2} \cdots T_{i_k}.$$

Let $\mathbb{F}_n^+$ be the unital free semigroup on n generators $g_1, \dots, g_n$ and the identity e. For each $\alpha \in \mathbb{F}_n^+$, define

$$e_\alpha := \begin{cases} e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_k}, & \text{if } \alpha = g_{i_1} g_{i_2} \cdots g_{i_k}, \\ 1, & \text{if } \alpha = e. \end{cases}$$

It is easy to see that $\{e_\alpha : \alpha \in \mathbb{F}_n^+\}$ is an orthonormal basis of $\mathcal{F}^2$. We also use $\mathbb{F}_n^+$ to denote arbitrary products of operators. If $T_1, \dots, T_n \in B(\mathcal{H})$, define

$$T_\alpha := \begin{cases} T_{i_1} T_{i_2} \cdots T_{i_k}, & \text{if } \alpha = g_{i_1} g_{i_2} \cdots g_{i_k}, \\ I_{\mathcal{H}}, & \text{if } \alpha = e. \end{cases}$$

The length of $\alpha \in \mathbb{F}_n^+$ is defined by $|\alpha| = k$, if $\alpha = g_{i_1} g_{i_2} \cdots g_{i_k}$, and $|\alpha| = 0$, if $\alpha = e$. For each $i = 1, \dots, n$, the left creation operator

$$S_i : \mathcal{F}^2 \to \mathcal{F}^2 \quad \text{is defined by} \quad S_i \psi = e_i \otimes \psi, \ \psi \in \mathcal{F}^2.$$

As in [Po4], the noncommutative disc algebra $\mathcal{A}_n$ is the norm closure of the algebra generated by $S_1, \ldots, S_n$ and $I_{\mathcal{F}^2}$, and the noncommutative analytic Toeplitz algebra F^∞ is the weak operator topology closure of $\mathcal{A}_n$ (see [Po5]).

Throughout this paper we consider a set $\{\omega_\alpha\}_{\alpha\in\mathbb{F}_n^+}$ of weights satisfying the following properties:

- (ω_1) $\omega_\alpha > 0$ for any $\alpha \in \mathbb{F}_n^+$;
- (ω_2) $\frac{\omega_{\alpha\beta\gamma}}{\omega_{\beta\gamma}} \leq \frac{\omega_{\alpha\beta}}{\omega_\beta}$ for any $\alpha, \beta, \gamma \in \mathbb{F}_n^+$, with $|\alpha| = 1$ and $|\gamma| = 1$;
- (ω_3) $\sum\limits_{\alpha\in\mathbb{F}_n^+} \frac{1}{\omega_\alpha}\lambda_\alpha S_\alpha$ is invertible in F^∞ for any $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{B}_n$, the open unit ball of $\mathbb{C}^n$.

Notice that if $\omega_\alpha = 1$, $\alpha \in \mathbb{F}_n^+$, these conditions are satisfied. Other examples will be considered in considered in Section 3. According to (ω_3), we have $(\sum\limits_{\alpha\in\mathbb{F}_n^+} \frac{1}{\omega_\alpha}\lambda_\alpha S_\alpha)^{-1} = \sum\limits_{\alpha\in\mathbb{F}_n^+} a_\alpha \lambda_\alpha S_\alpha$ for some constants $a_\alpha \in \mathbb{C}$. Hence, we deduce $a_e = \omega_e$, and for any $\gamma \in \mathbb{F}_n^+\backslash\{e\}$,

$$\sum_{\alpha\beta=\gamma} \frac{a_\beta}{\omega_\alpha} = 0, \tag{1.1}$$

$$\sum_{\beta\alpha=\gamma} \frac{a_\beta}{\omega_\alpha} = 0. \tag{1.2}$$

The following lemma is an extension of a result from [Q] to our setting.

Lemma 1.1. *$a_e > 0$ and $a_\alpha \leq 0$ for any $\alpha \in \mathbb{F}_n^+\backslash\{e\}$.*

Proof. It is clear that $a_e > 0$. If $\gamma \in \mathbb{F}_n^+$, $|\gamma| = 1$, then (1.2) shows that $a_\gamma = -\omega_e \frac{a_e}{\omega_\gamma} \leq 0$. Now we prove by induction. Assume that

$$a_\alpha \leq 0 \quad \text{for any } \alpha \in \mathbb{F}_n^+ \text{ with } 1 \leq |\alpha| \leq k. \tag{1.3}$$

Let $\gamma = \beta\gamma_{k+1}$ with $\beta, \gamma_{k+1} \in \mathbb{F}_n^+$, $|\beta| = k$, and $|\gamma_{k+1}| = 1$. According to (1.2), (ω_2), and (1.3), we infer that

$$a_e = -\sum_{\substack{\alpha\sigma=\beta\\|\alpha|\geq 1}} \frac{\omega_\beta}{\omega_\sigma} a_\alpha \geq -\sum_{\substack{\alpha\epsilon=\gamma,\\|\alpha|\geq 1, |\epsilon|\geq 1}} \frac{\omega_\gamma}{\omega_\epsilon} a_\alpha.$$

Hence, we deduce that

$$\sum_{\substack{\alpha\epsilon=\gamma,\\|\epsilon|\geq 1}} \frac{a_\alpha}{\omega_\epsilon} \geq 0. \tag{1.4}$$

According to (1.2), we have $\sum_{\alpha\epsilon=\gamma}\frac{a_\alpha}{\omega_\epsilon}=0$. Using (1.4), we obtain $a_\gamma\leq 0$. This completes the proof. ■

Notice that for any $\gamma\in\mathbb{F}_n^+$ and $0\leq p_1\leq p_2\leq|\gamma|$,

$$\sum_{\substack{\alpha\beta=\gamma\\|\alpha|\leq p_2}}\frac{a_\alpha}{\omega_\beta}\leq\sum_{\substack{\alpha\beta=\gamma\\|\alpha|\leq p_1}}\frac{a_\alpha}{\omega_\beta}. \tag{1.5}$$

Hence, we deduce

$$0\leq\omega_\gamma\left[\sum_{\substack{\alpha\beta=\gamma\\|\alpha|\leq p}}\frac{a_\alpha}{\omega_\beta}\right]\leq a_e \tag{1.6}$$

for any $0\leq p\leq|\gamma|$. On the other hand, since a_e is the only positive number among the a_α's, we obtain

$$\sum_{\substack{\alpha\beta=\gamma\\|\beta|\leq p}}\frac{a_\alpha}{\omega_\beta}\leq 0\quad\text{and}\quad\sum_{\substack{\beta\alpha=\gamma\\|\beta|\leq p}}\frac{a_\alpha}{\omega_\beta}\leq 0 \tag{1.7}$$

for any $p<|\gamma|$.

For each $i=1,\ldots,n$, the weighted left creation operator $W_i:\mathcal{F}^2\to\mathcal{F}^2$ is defined by $W_i=W_{g_i}:=S_iD_i$ where D_i is the diagonal operator acting on $\mathcal{F}^2$ by $D_i(e_\alpha):=\mu_{g_i,\alpha}e_\alpha$, and $\mu_{g_i,\alpha}:=\frac{\sqrt{\omega_{g_i\alpha}}}{\sqrt{\omega_\alpha}}$ for any $\alpha\in\mathbb{F}_n^+$. According to (ω_2), $\omega_e\omega_{g_i\alpha}\leq\omega_{g_i}\omega_\alpha$. Hence, $\|W_i\|=\|D_i\|=\frac{\sqrt{\omega_{g_i}}}{\sqrt{\omega_e}}$. Notice also that $W_\beta e_\alpha=\frac{\sqrt{\omega_{\beta\alpha}}}{\sqrt{\omega_\alpha}}e_{\beta\alpha}$ and

$$W_\beta^*e_\gamma=\begin{cases}\frac{\sqrt{\omega_\gamma}}{\sqrt{\omega_\alpha}}e_\alpha, & \text{if } \gamma=\beta\alpha \text{ for some } \gamma\in\mathbb{F}_n^+,\\ 0, & \text{otherwise.}\end{cases} \tag{1.8}$$

We define the weighted noncommutative disc algebra $\mathcal{A}_n(\omega_\alpha)$ to be the norm closure of all polynomials in $W_1,\ldots,W_n$ and the identity on $\mathcal{F}^2$.

Let $\varphi(W_1,\ldots,W_n)=\sum_{\alpha\in\mathbb{F}_n^+}b_\alpha W_\alpha$ be a formal sum such that

$$\sum_{\alpha\in\mathbb{F}_n^+}b_\alpha W_\alpha e_0\in\mathcal{F}^2,\quad\text{i.e.,}\quad\sum_{\alpha\in\mathbb{F}_n^+}|b_\alpha|^2\omega_\alpha<\infty.$$

According to the relation (ω_2), we have $\sum_{\alpha\in\mathbb{F}_n^+}b_\alpha W_\alpha p\in\mathcal{F}^2$ for any $p\in\mathcal{P}$. Assume that $\sup\|\sum_{\alpha\in\mathbb{F}_n^+}b_\alpha W_\alpha p\|_{\mathcal{F}^2}<\infty$, where the supremum is taken over all polynomials $p\in\mathcal{P}$ with $\|p\|_{\mathcal{F}^2}\leq 1$. Then there is a unique bounded operator on $\mathcal{F}^2$, denoted by $\varphi(W_1,\ldots,W_n)$, such that

$$\varphi(W_1,\ldots,W_n)p=\sum_{\alpha\in\mathbb{F}_n^+}b_\alpha W_\alpha p$$

for any $p \in \mathcal{P}$. The set $\{\varphi(W_1, \ldots, W_n)\}$ of all the operators satisfying the above-mentioned properties is denoted by $F^\infty(\omega_\alpha)$. It is clear that $F^\infty(\omega_\alpha)$ is an operator algebra, and if $\omega_\alpha = 1,\ \alpha \in \mathbb{F}_n^+$, then it coincides with the noncommutative analytic Toeplitz algebra F^∞ (see [Po4]). We will prove in Section 2 that $F^\infty(\omega_\alpha)$ is the WOT-closure (resp. w^*- closure) of all polynomials in $W_1, \ldots, W_n$ and the identity.

Alternatively, we define the weighted full Fock space $\mathcal{F}^2(\omega_\alpha)$ to be the Hilbert space with a complete orthogonal system $\{f_\alpha : \alpha \in \mathbb{F}_n^+\}$ and the weights $\langle f_\alpha, f_\alpha \rangle = \omega_\alpha$ for any $\alpha \in \mathbb{F}_n^+$. Let $U : \mathcal{F}^2 \to \mathcal{F}^2(\omega_\alpha)$ be the unitary operator defined by $U(e_\alpha) = \frac{1}{\sqrt{\omega_\alpha}} f_\alpha,\ \alpha \in \mathbb{F}_n^+$, and set $V_\beta := U W_\beta U^{-1}$ for any $\beta \in \mathbb{F}_n^+$. Notice that $V_\beta f_\alpha = f_{\beta\alpha}$ for any $\alpha, \beta \in \mathbb{F}_n^+$. We denote by $\mathcal{P}(\omega_\alpha)$ the set of all *polynomials* of the form $p = \sum_{\text{finite}} b_\alpha f_\alpha$, with the multiplication induced by $f_\alpha \otimes f_\beta := f_{\alpha\beta}$. Let $\mathcal{F}^\infty(\omega_\alpha)$ be the set of those $\varphi \in \mathcal{F}^2(\omega_\alpha)$ such that

$$\|\varphi\|_\infty := \sup\{\|\varphi \otimes p\|_{\mathcal{F}^2(\omega_\alpha)} : p \in \mathcal{P}(\omega_\alpha), \|p\|_{\mathcal{F}^2(\omega_\alpha)} \le 1\} < \infty.$$

It is clear that if $\varphi, \psi \in \mathcal{F}^\infty(\omega_\alpha)$, then $\varphi \otimes \psi \in \mathcal{F}^\infty(\omega_\alpha)$. The norm $\|\varphi\|_\infty$ coincides with the operator norm of $\varphi(V_1, \ldots, V_n) := U\varphi(W_1, \ldots, W_n)U^{-1}$. Therefore, we can identify $\mathcal{F}^\infty(\omega_\alpha)$ with $UF^\infty(\omega_\alpha)U^{-1}$, and $\varphi \in \mathcal{F}^\infty(\omega_\alpha)$ can be viewed as being an element in $F^\infty(\omega_\alpha)$ and conversely.

We refer to [Ar1], [P], and [Pi] for results on completely bounded maps and operator theory.

2. Noncommutative Poisson transforms

Let $W = [W_1, \ldots, W_n]$ be the set of weighted left creation operators on the full Fock space $\mathcal{F}^2$, as defined in Section 1. Following [Po7], we define in a similar manner a noncommutative Poisson transform on $C^*(W_1, \ldots, W_n)$, the C^*-algebra generated by $W_1, \ldots, W_n$ and the identity.

According to (1.8), one can easily check that for every $\alpha, \beta \in \mathbb{F}_n^+$,

$$W_\beta W_\beta^* e_\alpha = \begin{cases} \frac{\omega_\alpha}{\omega_\gamma} e_\alpha, & \text{if } \alpha = \beta\gamma \text{ for some } \gamma \in \mathbb{F}_n^+, \\ 0, & \text{otherwise.} \end{cases}$$

Using this formula we can deduce that for each $N \ge 0$

$$\sum_{|\beta| \le N} a_\beta W_\beta W_\beta^* e_\alpha = b_{\alpha,N} e_\alpha, \text{ where } \quad b_{\alpha,N} = \omega_\alpha \sum_{\substack{\gamma\beta = \alpha \\ |\gamma| \le N}} \frac{a_\gamma}{\omega_\beta}.$$

From (1.2) and (1.6) we infer that $b_{e,N} = a_e$, that $b_{\alpha,N} = 0$ if $1 \le |\alpha| \le N$, and that $0 \le b_{\alpha,N} \le a_e$ if $|\alpha| > N$. Using these relations, Lemma 1.1, and (1.5), we see that $\left\{\sum_{|\beta|\leq N} a_\beta W_\beta W_\beta^*\right\}_N$ is a non-increasing sequence of non-negative operators which converges in the strong operator topology to

$$\Delta(W, W^*) := \sum_{\beta\in\mathbb{F}_n^+} a_\beta W_\beta W_\beta^* = a_e P_0,$$

where P_0 is the orthogonal projection onto the span of the constant function 1. Since

$$P_0 S_\alpha^* e_\gamma = \begin{cases} \frac{\sqrt{\omega_\beta}}{\sqrt{\omega_e}} e_0, & \text{if } \beta = \gamma, \\ 0, & \text{otherwise,} \end{cases}$$

and using (1.2), it is easy to see that $\sum_{\alpha\in\mathbb{F}_n^+} \frac{1}{\omega_\alpha} W_\alpha a_e P_0 W_\alpha^* e_\gamma = e_\gamma$. Therefore,

$$\sum_{\alpha\in\mathbb{F}_n^+} \frac{1}{\omega_\alpha} W_\alpha \Delta(W, W^*) W_\alpha^* = I_{\mathcal{F}^2},$$

where the convergence is in the strong operator topology.

Let $T = [T_1, \ldots, T_n]$ be an n-tuple of operators on a Hilbert space $\mathcal{H}$. Suppose that for each $N \geq 0$

$$\sum_{|\beta|\leq N} a_\beta T_\beta T_\beta^* \geq 0. \tag{2.1}$$

Since $a_\beta \leq 0$ whenever $|\beta| \geq 1$, it follows that $\left\{\sum_{|\beta|\leq N} a_\beta T_\beta T_\beta^*\right\}_{N=1}^\infty$ is a non-increasing sequence of non-negative operators in $B(\mathcal{H})$ which converges in the strong operator topology to

$$\Delta(T, T^*) := \sum_{\beta\in\mathbb{F}_n^+} a_\beta T_\beta T_\beta^* \geq 0. \tag{P}$$

For each $N \geq 0$ and $h \in \mathcal{H}$,

$$\begin{aligned}
&\sum_{|\alpha|\leq N} \frac{1}{\omega_\alpha} \left\langle T_\alpha \Delta(T, T^*) T_\alpha^* h, h\right\rangle \\
&\quad = \sum_{|\alpha|\leq N} \sum_{\beta\in\mathbb{F}_n^+} \frac{a_\beta}{\omega_\alpha} \left\|T_{\alpha\beta}^* h\right\|^2 \\
&\quad = \sum_{\gamma\in\mathbb{F}_n^+} \left[\sum_{\substack{\alpha\beta=\gamma \\ |\alpha|\leq N}} \frac{a_\beta}{\omega_\alpha}\right] \left\|T_\gamma^* h\right\|^2 \\
&\quad = \|h\|^2 + \sum_{1\leq|\gamma|\leq N} \left[\sum_{\alpha\beta=\gamma} \frac{a_\beta}{\omega_\alpha}\right] \left\|T_\gamma^* h\right\|^2 + \sum_{|\gamma|>N} \left[\sum_{\substack{\alpha\beta=\gamma \\ |\alpha|\leq N}} \frac{a_\beta}{\omega_\alpha}\right] \left\|T_\gamma^* h\right\|^2.
\end{aligned}$$

It follows from (1.1) that $\sum_{\alpha\beta=\gamma}\frac{a_\beta}{\omega_\alpha}=0$. On the other hand, using (1.2) and (1.3), we deduce that $\sum_{\substack{\alpha\beta=\gamma\\|\alpha|\leq N}}\frac{a_\beta}{\omega_\alpha}\leq 0$ whenever $|\gamma|>N$. Hence,

$$0\leq\sum_{|\alpha|\leq N}\frac{1}{\omega_\alpha}T_\alpha\Delta(T,T^*)T_\alpha^*=I_{\mathcal{H}}+\sum_{\gamma>N}\Bigg[\sum_{\substack{\alpha\beta=\gamma\\|\alpha|\leq N}}\frac{a_\beta}{\omega_\alpha}\Bigg]T_\gamma T_\gamma^*\leq I_{\mathcal{H}}.$$

Therefore,

$$\sum_{\alpha\in\mathbb{F}_n^+}\frac{1}{\omega_\alpha}T_\alpha\Delta(T,T^*)T_\alpha^*=I_{\mathcal{H}} \tag{2.2}$$

in the strong operator topology if and only if

$$\mathrm{SOT}-\lim_{N\to\infty}\sum_{|\gamma|>N}\Bigg[\sum_{\substack{\alpha\beta=\gamma\\|\alpha|\leq N}}\frac{a_\beta}{\omega_\alpha}\Bigg]T_\gamma T_\gamma^*=0. \tag{C_0}$$

For example, if $T=[T_1,\ldots,T_n]$ is any n-tuple that satisfies (P), then $rT:=[rT_1,\ldots,rT_n]$ satisfies (C_0) for every $0<r<1$.

Assume that $T=[T_1,\ldots,T_n]$ are operators in $B(\mathcal{H})$ satisfying (P) and (C_0). Define the *defect* operator

$$D:=\left(\Delta(T,T^*)\right)^{\frac{1}{2}}.$$

The Poisson kernel K_T associated to $T=[T_1,\ldots,T_n]$ is the linear map

$$K_T:\mathcal{H}\to\mathcal{F}^2(H_n)\otimes\mathcal{H}\qquad\text{defined by}\qquad K_Th=\sum_{\alpha\in\mathbb{F}_n^+}\frac{e_\alpha}{\sqrt{\omega_\alpha}}\otimes DT_\alpha^*h.$$

It follows from (2.2) that K_T is an isometry. Moreover, it is easy to check that for every $\alpha\in\mathbb{F}_n^+$, $(W_\alpha^*\otimes I_{\mathcal{H}})K_Th=K_T(T_\alpha^*h)$. Hence, for every $\alpha,\beta\in\mathbb{F}_n^+$,

$$K_T^*\big[W_\alpha W_\beta^*\otimes I\big]K_T=T_\alpha T_\beta^*. \tag{2.3}$$

The map $\Psi_T:B(\mathcal{F}^2)\to B(\mathcal{H})$ defined by $\Psi_T(A)=K_T^*[A\otimes I]K_T$ is clearly unital, completely contractive (hence, completely positive), and w^*-continuous. Moreover, for each $\alpha,\beta\in\mathbb{F}_n^+$, $\Psi_T(W_\alpha W_\beta^*)=T_\alpha T_\beta^*$.

Proposition 2.1. *The set $\mathcal{P}(\omega_\alpha)$ of all polynomials in $W_1,\ldots,W_n$ and the identity is w^*-dense in $F^\infty(\omega_\alpha)$.*

Proof. If $\varphi(W_1,\ldots,W_n)=\sum_{\alpha\in\mathbb{F}_n^+}b_\alpha W_\alpha$ is in $F^\infty(\omega_\alpha)$ and $0<r<1$, then

$$\varphi_r(W_1,\ldots,W_n):=\sum_{\alpha\in\mathbb{F}_n^+}b_\alpha r^{|\alpha|}W_\alpha$$

converges in norm, so that $\varphi_r(W_1, \ldots, W_n) \in \mathcal{A}_n(\omega_\alpha)$. Setting $T := [rW_1, \ldots, rW_n]$, it is clear that T satisfies (P) and (C_0). Using (2.3), one can easily check that $\Psi_T(\varphi(W_1, \ldots, W_n)) = \varphi(rW_1, \ldots, rW_n)$ and therefore,

$$\|\varphi_r(W_1, \ldots, W_n)\| \leq \|\varphi(W_1, \ldots, W_n)\|$$

for any $0 < r < 1$. One can also easily check that

$$\varphi(W_1, \ldots, W_n) = \text{SOT} - \lim_{r \to 1} \varphi_r(W_1, \ldots, W_n)$$

(see for example Proposition 4.5 of [Po5]). Since the set $\mathcal{P}(\omega_\alpha)$ of all polynomials in $W_1, \ldots, W_n$ and the identity is norm dense in $\mathcal{A}_n(\omega_\alpha)$, we infer that $\mathcal{P}(\omega_\alpha)$ is WOT-dense in $F^\infty(\omega_\alpha)$. Since the w^* and WOT topologies coincide on bounded sets, the result follows. ∎

Let $T = [T_1, \ldots, T_n]$ be an n-tuple of operators satisfying (P) and (C_0). Notice that for any $\varphi(W_1, \ldots, W_n) \in F^\infty(\omega_\alpha)$, $\varphi_r(T_1, \ldots, T_n) := \sum_{\alpha \in \mathbb{F}_n^+} b_\alpha r^{|\alpha|} T_\alpha$ converges in norm. Similarly to Theorem 4.3 of [Po5], one can show that

$$\varphi(T_1, \ldots, T_n) := \text{SOT} - \lim_{r \to 1} \varphi_r(T_1, \ldots, T_n)$$

exists. Notice that $\Psi_T(\varphi(W_1, \ldots, W_n)) = \varphi(T_1, \ldots, T_n)$ for any $\varphi(W_1, \ldots, W_n)$ in $F^\infty(\omega_\alpha)$. Therefore, the restriction of Ψ_T to $F^\infty(\omega_\alpha)$, which is also denoted by Ψ_T, provides a WOT-continuous $F^\infty(\omega_\alpha)$-functional calculus for n-tuples of operators $T = [T_1, \ldots, T_n]$ satisfying properties (P) and (C_0). We have proved the following.

Proposition 2.2. *Let $T = [T_1, \ldots, T_n]$ be an n-tuple of operators satisfying property (P) and (C_0). The Poisson transform $\Psi_T : F^\infty(\omega_\alpha) \to B(\mathcal{H})$ defined by*

$$\Psi_T(\varphi(W_1, \ldots, W_n)) = K_T^*(\varphi(W_1, \ldots, W_n) \otimes I)K_T \tag{2.4}$$

is WOT-continuous, completely contractive homomorphism and

$$\Psi_T(\varphi(W_1, \ldots, W_n)) = \varphi(T_1, \ldots, T_n)$$

for every $\varphi(W_1, \ldots, W_n) \in F^\infty(\omega_\alpha)$.

Notice that when $\omega_\alpha = 1$ we find again the F^∞-functional calculus for row contractions of class C_0 (see [Po5]).

Suppose now that $T = [T_1, \ldots, T_n]$ is an n-tuple of operators satisfying only the property (P). For each $0 < r < 1$, let $K_r = K_{rT}$ be the Poisson kernel associated to $rT := [rT_1, \dots rT_n]$, which clearly satisfies (C_0). Let $C^*(W_1, \dots, W_n)$ be the C^*-algebra generated by $W_1, \dots, W_n$. Following [Po7], one can similarly prove that for any $f(W_1, \dots, W_n) \in C^*(W_1, \dots, W_n)$, $\lim_{r\to 1} K_r^*[f(W_1, \dots, W_n) \otimes I]K_r$ exists in the uniform topology of $B(\mathcal{H})$.

The Poisson transform associated to $T = [T_1, \dots, T_n]$ is the linear map $\Phi_T : C^*(W_1, \dots, W_n) \to B(\mathcal{H})$ defined by

$$\Phi_T(f(W_1, \dots, W_n)) = \lim_{r\to 1} K_r^*[f(W_1, \dots, W_n) \otimes I]K_r. \tag{2.5}$$

Proposition 2.3. *If $T = [T_1, \dots, T_n]$ is an n-tuple of operators satisfying the property (P), then the Poisson transform Φ_T is a unital, completely contractive linear map such that $\Phi_T(W_\alpha W_\beta^*) = T_\alpha T_\beta^*$ for every $\alpha, \beta \in \mathbb{F}_n^+$.*

Moreover, one can easily see now that a sequence of operators $T = [T_1, \dots, T_n]$ has property (P) if and only if there is a completely contractive linear map $\psi : C^*(W_1, \dots, W_n) \to B(\mathcal{H})$ such that $\psi(I) = I_{\mathcal{H}}$ and $\psi(W_\alpha W_\beta^*) = T_\alpha T_\beta^*$ for every $\alpha, \beta \in \mathbb{F}_n^+$.

Remark 2.4. *When $\omega_\alpha = 1, \quad \alpha \in \mathbb{F}_n^+$, we have $a_e = 1, a_\gamma = -1$ if $|\gamma| = 1$, and $a_\gamma = 0$ if $|\gamma| > 1$. In this case, the corresponding Poisson transform coincides with the one developed in* [Po7] *in the particular case when $T = [T_1, \dots, T_n]$ is a row contraction.*

Let $J \subset F^\infty(\omega_\alpha)$ be a w^*-closed ideal of $F^\infty(\omega_\alpha)$. Define $\mathcal{M}_J := \overline{J\mathcal{F}^2}$, and $\mathcal{N}_J := \mathcal{F}^2 \ominus \mathcal{M}_J$. Since J is a right ideal,

$$\{\varphi(W_1, \dots, W_n)e_0 : \varphi \in J\} \tag{2.6}$$

is dense in $\mathcal{M}_J$; and since J is a left ideal, $\mathcal{N}_J$ is invariant under $W_1^*, \dots, W_n^*$. For each $k \le n$, let $B_k = P_{\mathcal{N}_J} W_{k|\mathcal{N}_J}$ be the compression of W_k to $\mathcal{N}_J$. Since $\mathcal{N}_J$ is semi invariant under the W_i's, we have that $B_\alpha = P_{\mathcal{N}_J} W_\alpha|_{\mathcal{N}_J}$ for any $\alpha \in \mathbb{F}_n^+$.

The following result is an extension of Theorem 3.7 from [APo2] to our setting.

Proposition 2.5. *Suppose that $T = [T_1, \dots, T_n]$ are operators in $B(\mathcal{H})$ that satisfy (P) and (C_0) and that $J \subset F^\infty(\omega_\alpha)$ is a WOT-closed ideal of $F^\infty(\omega_\alpha)$ such that for each $\varphi \in J$, $\varphi(T_1, \dots, T_n) = 0$. Then, for every $h \in \mathcal{H}$, $K_T h \in \mathcal{N}_J \otimes \mathcal{H}$. Moreover, the map $\Phi_{T,J} : B(\mathcal{N}_J) \to B(\mathcal{H})$, defined by $\Phi_{T,J}(A) = K_T^*(A \otimes I_{\mathcal{H}})K_T$, is w^*-continuous, unital, completely positive, and satisfies $\Phi_{T,J}(B_\alpha B_\beta^*) = T_\alpha T_\beta^*$ for every $\alpha, \beta \in \mathbb{F}_n^+$, where $B_\alpha = P_{\mathcal{N}_J} W_{\alpha|\mathcal{N}_J}$ is the compression of W_α to $\mathcal{N}_J$.*

Proof. For any polynomial $p = \sum_\beta b_\beta e_\beta$ in $F^2(H_n)$ and $h, h' \in \mathcal{H}$, we have

$$\begin{aligned}\langle K_T h, p(W_1, \ldots, W_n)e_0 \otimes h'\rangle &= \frac{1}{\sqrt{\omega_e}} \sum_\alpha \bar{b}_\alpha \langle DT_\alpha^* h, h'\rangle \\ &= \frac{1}{\sqrt{\omega_e}} \sum_\alpha \bar{b}_\alpha \langle h, T_\alpha D h'\rangle = \frac{1}{\sqrt{\omega_e}} \langle h, p(T_1, \ldots, T_d) D h'\rangle.\end{aligned}$$

Since the $F^\infty(\omega_\alpha)$-functional calculus is WOT-continuous and $\mathcal{P}(\omega_\alpha)$ is WOT-dense in $F^\infty(\omega_\alpha)$ we infer that

$$\langle K_T h, \varphi(W_1, \ldots, W_n)e_0 \otimes h'\rangle = \frac{1}{\sqrt{\omega_e}} \langle h, \varphi(T_1, \ldots, T_n) D h'\rangle$$

for every $\varphi(W_1, \ldots, W_n) \in F^\infty(\omega_\alpha)$. Hence, $\langle K_T h, \varphi(W_1, \ldots, W_n)e_0 \otimes h'\rangle = 0$ whenever $\varphi \in J$. According to (2.6), we conclude that $K_T h \in (\mathcal{M}_J \otimes \mathcal{H})^\perp = \mathcal{N}_J \otimes \mathcal{H}$ and therefore, $K_T = (P_{\mathcal{N}_J} \otimes I_{\mathcal{H}})K_T$. Since $\mathcal{N}_J \subset \mathcal{F}^2$ is invariant under $W_1^*, \ldots, W_n^*$ and $B_\alpha B_\beta^* = P_{\mathcal{N}_J} W_\alpha W_\beta^*|_{\mathcal{N}_J}$ for $\alpha, \beta \in \mathbb{F}_n^+$, it follows from (2.3) that

$$\begin{aligned}T_\alpha T_\beta^* &= K_T^*(W_\alpha W_\beta^* \otimes I_{\mathcal{H}})K_T = K_T^*(P_{\mathcal{N}_J} \otimes I_{\mathcal{H}})(W_\alpha W_\beta^* \otimes I_{\mathcal{H}})(P_{\mathcal{N}_J} \otimes I_{\mathcal{H}})K_T \\ &= K_T^*(P_{\mathcal{N}_J} W_\alpha W_\beta^* P_{\mathcal{N}_J} \otimes I_{\mathcal{H}})K_T = K_T^*(B_\alpha B_\beta^* \otimes I_{\mathcal{H}})K_T\end{aligned}$$

for every $\alpha, \beta \in \mathbb{F}_n^+$. This complets the proof. ■

Let us remark that if $T = [T_1, \ldots, T_n]$ has property (P) and its Poisson kernel K_r takes values in $\mathcal{N}_J \otimes \mathcal{H}$ for every $0 < r < 1$, then there is a unital, completely contractive linear map $\Phi : C^*(B_1, \ldots, B_n) \to B(\mathcal{H})$ satisfying $\Phi(B_\alpha B_\beta^*) = T_\alpha T_\beta^*$ for all $\alpha, \beta \in \mathbb{F}_n^+$. The proof is similar to Remark 3.2 from [APo2], so we omit it.

3. Isometric representations of quotient algebras

In this section we obtain representation theorems for quotients of $F^\infty(\omega_\alpha)$ and the space of multipliers of some weighted Hardy spaces on the unit ball of $\mathbb{C}^n$. We need the following result from [APo2].

Proposition 3.1. *Let A be a w^*-closed unital subalgebra of $B(\mathcal{H})$ and let $J \subset A$ be a w^*-closed, 2-sided ideal. Then there exists $\mathcal{E} \subset \mathcal{H} \otimes \ell_2$ such that the map $\widehat{\Psi} : A/J \to B(\mathcal{E})$ defined by $\widehat{\Psi}(a + J) = P_{\mathcal{E}}(a \otimes I_{\ell_2})|_{\mathcal{E}}$ is a w^*-continuous, completely isometric representation.*

The map that sends $Y \in B(\mathcal{F}^2)$ to $Y \otimes I_{\ell_2} \in B(\mathcal{F}^2 \otimes \ell_2)$ is unital, completely positive, and WOT-continuous. Then it follows that $[W_1 \otimes I_{\ell_2}, \ldots, W_n \otimes I_{\ell_2}]$ satisfies (P) and (C_0).

Theorem 3.2. *Let $J \subset F^\infty(\omega_\alpha)$ be a w^*-closed ideal of $F^\infty(\omega_\alpha)$. Then the map $\Psi : F^\infty(\omega_\alpha)/J \to B(\mathcal{N}_J)$ defined by*

$$\Psi(\varphi(W_1,\dots,W_n)+J) = P_{\mathcal{N}_J}\varphi(W_1,\dots,W_n)|_{\mathcal{N}_J}$$

is a w^*-continuous, completely isometric representation.

Proof. We will first show that Ψ is a w^*-continuous map. Let $\varphi_\lambda + J$ be a net that converges to $\varphi + J$ in the w^*-topology of $F^\infty(\omega_\alpha)/J$ and let $X \in c_1(\mathcal{N}_J)$ be an operator in the trace class of $\mathcal{N}_J$. Since $tr(\eta X) = 0$ for every $\eta \in J$, X induces a w^*-continuous linear functional on $F^\infty(\omega_\alpha)/J$, given by $\langle \phi + J, X\rangle := tr(P_{\mathcal{N}_J}\phi|_{\mathcal{N}_J}X)$, $\phi \in F^\infty(\omega_\alpha)$. Hence, we check that

$$tr(\Psi(\varphi+J)X) = \langle \varphi + J, X\rangle = \lim_\lambda \langle \varphi_\lambda + J, X\rangle = \lim_\lambda tr(\Psi(\varphi_\lambda + J)X).$$

This shows that Ψ is w^*-continuous. Applying Proposition 3.1 to $F^\infty(\omega_\alpha)/J$, we find a w^*-continuous, completely isometric representation $\widehat{\Psi} : F^\infty(\omega_\alpha)/J \to B(\mathcal{E})$. For each $k = 1,\dots,n$, define

$$T_k := \widehat{\Psi}(W_k + J) = P_{\mathcal{E}} W_k \otimes I_{\ell_2}|_{\mathcal{E}}.$$

Then, for $h \in \mathcal{E}$, $\langle T_\beta T_\beta^* h, h\rangle = \|T_\beta^* h\|^2 = \|P_{\mathcal{E}}(W_\beta^* \otimes I_{\ell_2})h\|^2 \le \|(W_\beta^* \otimes I_{\ell_2})h\|^2$. This shows that

$$T_\beta T_\beta^* \le P_{\mathcal{E}}(W_\beta \otimes I_{\ell_2})(W_\beta^* \otimes I_{\ell_2})|_{\mathcal{E}} \quad \text{for every} \quad \beta \in \mathbb{F}_n^+. \tag{3.1}$$

When $\beta = e$, the two sides of (3.1) are equal to each other, and when $\beta \in \mathbb{F}_n^+\backslash\{e\}$, $a_\beta \le 0$. Hence, for every $N \ge 0$,

$$\sum_{|\beta|\le N} a_\beta T_\beta T_\beta^* \ge P_{\mathcal{E}}\Bigg(\sum_{|\beta|\le N} a_\beta (W_\beta \otimes I_{\ell_2})(W_\beta^* \otimes I_{\ell_2})\Bigg)|_{\mathcal{E}} \ge 0,$$

which shows that $T = [T_1,\dots,T_n]$ satisfies property (P). It follows from (1.7) that $\sum_{\substack{\alpha\beta=\gamma\\ |\alpha|\le N}} \frac{a_\beta}{\omega_\alpha} \le 0$ whenever $|\gamma| > N$. Hence,

$$P_{\mathcal{E}} \sum_{|\gamma|>N}\Bigg[\sum_{\substack{\alpha\beta=\gamma\\ |\alpha|\le N}} \frac{a_\beta}{\omega_\alpha}\Bigg](W_\gamma \otimes I_{\ell_2})(W_\gamma^* \otimes I_{\ell_2})|_{\mathcal{E}} \le \sum_{|\gamma|>N}\Bigg[\sum_{\substack{\alpha\beta=\gamma\\ |\alpha|\le N}} \frac{a_\beta}{\omega_\alpha}\Bigg] T_\gamma T_\gamma^* \le 0.$$

Since $[W_1 \otimes I_{\ell_2},\dots,W_n \otimes I_{\ell_2}]$ has property (C_0) it is easy to see that $T = [T_1,\dots,T_n]$ has the same property.

For any polynomial $p \in \mathcal{P}(\omega_\alpha)$, $\widehat{\Psi}(p+J) = p(T_1,\dots,T_n)$. Since $\widehat{\Psi}$ is w^*-continuous, $\widehat{\Psi}(\varphi + J) = \varphi(T_1,\dots,T_n)$ for every $\varphi \in F^\infty(\omega_\alpha)$. Hence, if $\varphi \in J$, it follows from Proposition 3.1 that $\|\varphi(T_1,\dots,T_n)\| = \|\widehat{\Psi}(\varphi+J)\| = \operatorname{dist}(\varphi, J) = 0$. By Proposition 2.5, there exists a w^*-continuous map $\Phi_{T,J} : B(\mathcal{N}_J) \to B(\mathcal{E})$ such that $\Phi_{T,J}(B_\alpha) = T_\alpha$ for every $\alpha \in \mathbb{F}_n^+$, where $B_\alpha = P_{\mathcal{N}_J} W_\alpha|_{\mathcal{N}_J}$ is the compression of W_α to $\mathcal{N}_J$. Then $\widehat{\Psi}(W_\alpha + J) = T_\alpha = \Phi_{T,J}(B_\alpha) = \Phi_{T,J} \circ \Psi(W_\alpha + J)$ for every $\alpha \in \mathbb{F}_n^+$. Using the w^*-continuity of the these maps, we obtain the following commutative diagram

$$\begin{tikzcd}
F^\infty(\omega_\alpha)/J \arrow[rr, "\widehat{\Psi}"] \arrow[dr, "\Psi"'] & & B(\mathcal{E}) \\
& B(\mathcal{N}_J) \arrow[ur, "\Phi_{T,J}"'] &
\end{tikzcd} .$$

Since $\Phi_{T,J}$ and Ψ are completely contractive, and since $\widehat{\Psi}$ is completely isometric, we conclude that Ψ is completely isometric. ■

Let $\mathcal{W}(B_1,\dots,B_n)$ be the w^*-closure of the algebra generated by $B_1,\dots,B_n$ and the identity (recall that $B_k = P_{\mathcal{N}_J} W_k|_{\mathcal{N}_J}$ for $k \le n$).

Proposition 3.3. $\mathcal{W}(B_1,\dots,B_n) = \Psi\big(F^\infty(\omega_\alpha)/J\big)$.

Proof. Since Ψ is a w^*-continuous isometric representation, $\Psi\big(F^\infty(\omega_\alpha)/J\big)$ is a w^*-closed subalgebra of $B(\mathcal{N}_J)$ (see [R, Theorem 4.14]) that contains $B_1,\dots,B_n$ and the identity. Then it follows that $\mathcal{W}(B_1,\dots,B_n) \subset \Psi\big(F^\infty(\omega_\alpha)/J\big)$.

Since the set $\{p(W_1,\dots,W_n) + J : p \in \mathcal{P}\}$ is w^*-dense in $F^\infty(\omega_\alpha)/J$ and $\Psi(p(W_1,\dots,W_n)+J) = p(B_1,\dots,B_n)$, we conclude that $\{p(B_1,\dots,B_n) : p \in \mathcal{P}\}$ is w^*-dense in $\Psi\big(F^\infty(\omega_\alpha)/J\big)$. Hence, $\Psi\big(F^\infty(\omega_\alpha)/J\big) \subset \mathcal{W}(B_1,\dots,B_n)$. ■

The proof of the following result is similar to [APo2, Theorem 4.5], so we will omit it.

Proposition 3.4. *Let $T = [T_1,\dots,T_n]$ be an n-tuple of operators satisfying property (P) and (C_0), and let*

$$\Psi_T : F^\infty(\omega_\alpha) \to B(\mathcal{H}), \quad \Psi_T(f(W_1,\dots,W_n)) = f(T_1,\dots,T_n),$$

be the $F^\infty(\omega_\alpha)$-functional calculus associated to T. If J is a WOT*-closed, 2-sided ideal of $F^\infty(\omega_\alpha)$ with $J \subset \operatorname{Ker}\Psi_T$, then the map*

$$\Psi_{T,J} : \mathcal{W}(B_1,\dots B_n) \to B(\mathcal{H}); \quad \Psi_{T,J}(f(B_1,\dots,B_n)) := f(T_1,\dots,T_n),$$

is a WOT-*continuous, completely contractive homomorphism. In particular, for any* $f(W_1,\dots,W_n)\in F^\infty(\omega_\alpha)$,

$$\|f(T_1,\dots,T_n)\|\le\|f(B_1,\dots,B_n)\|=\operatorname{dist}(f(W_1,\dots,W_n),J).$$

We will now describe the quotients of $\mathcal{W}(B_1,\dots,B_n)$. Let $I\subset\mathcal{W}(B_1,\dots,B_n)$ be a w^*-closed, 2-sided ideal of $\mathcal{W}(B_1,\dots,B_n)$. Then $\Psi^{-1}(I)=\overline{I}/J$, where

$$\overline{I}=\{\varphi(W_1,\dots,W_n)\in F^\infty(\omega_\alpha):\varphi(B_1,\dots,B_n)\in I\}$$

is a w^*-closed, 2-sided ideal of $F^\infty(\omega_\alpha)$ that contains J. For each $k\le n$, let $D_k=P_{\mathcal{N}_{\overline{I}}}W_k|_{\mathcal{N}_{\overline{I}}}$ be the compression of W_k to $\mathcal{N}_{\overline{I}}$.

Proposition 3.5. *The map* $\Phi:F^\infty(\omega_\alpha)/\overline{I}\to\mathcal{W}(B_1,\dots,B_n)/I$ *defined by*

$$\Phi\big(\varphi(W_1,\dots,W_n)+\overline{I}\big):=\varphi(B_1,\dots,B_n)+I$$

is completely isometric, w^**-continuous, and onto. Therefore, by Theorem* 3.2*,* $\mathcal{W}(B_1,\dots,B_n)/I$ *is completely isometric to* $\mathcal{W}(D_1,\dots,D_n)$.

Proof. It is easy to see that Φ is a well defined w^*-continuous operator. Define $Q:F^\infty(\omega_\alpha)/J\to F^\infty(\omega_\alpha)/\overline{I}$ and $\pi:\mathcal{W}(B_1,\dots,B_n)\to\mathcal{W}(B_1,\dots,B_n)/I$ by $Q(\varphi+J)=\varphi+\overline{I}$ and $\pi(\varphi(B_1,\dots,B_n))=\varphi(B_1,\dots,B_n)+I$. Consider the following commutative diagram

$$\begin{tikzcd}
F^\infty(\omega_\alpha)/J \arrow[r, "\Psi"] \arrow[d, "Q"'] & \mathcal{W}(B_1,\dots,B_n) \arrow[r, "\Psi^{-1}"] \arrow[d, "\pi"'] & F^\infty(\omega_\alpha)/J \arrow[d, "Q"'] \\
F^\infty(\omega_\alpha)/\overline{I} \arrow[r, "\Phi"] & \mathcal{W}(B_1,\dots,B_n)/I \arrow[r, "\Phi^{-1}"] & F^\infty(\omega_\alpha)/\overline{I}.
\end{tikzcd}$$

Since Q and π are completely contractive, and Ψ and Ψ^{-1} are completely isometric, it follows that Φ and Φ^{-1} are completely contractive. ■

Now, let us consider some consequences of Theorem 3.2. When $n=1$ and $\omega_\alpha=1$ for every $\alpha\in\mathbb{F}_n^+$, $\mathcal{F}^2(\omega_\alpha)$ is the classical Hardy space H^2 and $F^\infty(\omega_\alpha)$ is the analytic Toeplitz algebra H^∞. In this case W_1 is the unilateral shift on H^2.

Corollary 3.6. [S] *Let* $J\subset H^\infty$ *be a* w^**-closed ideal of* H^∞*. Then the map* $\Psi:H^\infty/J\to B(\mathcal{N}_J)$ *defined by* $\Psi(\varphi+J)=P_{\mathcal{N}_J}\varphi(W_1)|_{\mathcal{N}_J}$ *is a* w^**-continuous completely isometric representation.*

When $n\ge2$ and $\omega_\alpha=1$ for every $\alpha\in\mathbb{F}_n^+$, $\mathcal{F}^2(\omega_\alpha)$ is the full Fock space $\mathcal{F}^2$ and $F^\infty(\omega_\alpha)$ is the non-commutative analytic Toeplitz algebra F^∞ (see [Po4]). The operators $W_1,\dots,W_n$ are now the left creation operators on the full Fock space $\mathcal{F}^2$.

Corollary 3.7. ([APo2] and [DP3]) *Let* $J \subset F^\infty$ *be a* w^**-closed ideal of* F^∞*. Then the map* $\Psi : F^\infty/J \to B(\mathcal{N}_J)$ *defined by* $\Psi(\varphi + J) = P_{\mathcal{N}_J}\varphi(W_1, \dots, W_n)|_{\mathcal{N}_J}$ *is a* w^**-continuous completely isometric representation.*

When $n = 1$ and $\omega_\alpha = |\alpha| + 1$ for every $\alpha \in \mathbb{F}_n^+$, $\mathcal{F}^2(\omega_\alpha)$ is the reproducing kernel Hilbert space with kernel on $\mathbb{D}$ given by

$$K(z, w) = \sum_{n=0}^{\infty} \frac{1}{\omega_n} z^n \bar{w}^n = \frac{1}{z\bar{w}} \log\left(\frac{1}{1 - z\bar{w}}\right).$$

$F^\infty(\omega_\alpha)$ is the Dirichlet algebra D, which is the space of multipliers of K.

Corollary 3.8. *Let* $J \subset D$ *be a* w^**-closed ideal of the Dirichlet algebra* D*. Then the map* $\Psi : D/J \to B(\mathcal{N}_J)$ *defined by* $\Psi(\varphi+J) = P_{\mathcal{N}_J}\varphi(W_1)|_{\mathcal{N}_J}$ *is a* w^**-continuous completely isometric representation.*

We shall consider now the particular case when $\omega_\alpha = \omega_{|\alpha|}$. Let $J_+ \subset F^\infty(\omega_\alpha)$ be the w^*-closed, 2-sided ideal of $F^\infty(\omega_n)$ generated by $\{W_iW_j - W_jW_i : i \neq j\}$. It is not hard to check that $\mathcal{N}_{J_+}$ is the symmetric Fock space $\mathcal{F}^2_+$. For $j \leq n$, let $B_j = P_{\mathcal{F}^2_+} W_j|_{\mathcal{F}^2_+}$ be the compression of W_j to $\mathcal{F}^2_+$, and let $\mathcal{W}(B_1, \dots, B_n)$ be the w^*-closure of the algebra generated by $B_1, \dots, B_n$ and the identity. It follows from Proposition 3.3 that $\mathcal{W}(B_1, \dots, B_n)$ can be identified with $F^\infty(\omega_\alpha)/J_+$.

Let $\mathbb{B}_n$ be the unit ball of $\mathbb{C}^n$. The weighted Hardy space $H^2(\mathbb{B}_n, \omega_k)$, where $\omega_k := \omega_\alpha$ if $|\alpha| = k$, is the Hilbert space with complete orthogonal system $\{z^\gamma : \gamma \in \mathbb{Z}^n, \gamma \geq 0\}$ and weights

$$\langle z^\gamma, z^\gamma \rangle = \omega_{|\gamma|} \frac{\gamma!}{|\gamma|!},$$

where, as usual, if $\gamma = (\gamma_1, \dots, \gamma_n) \in \mathbb{Z}^n$ and $\gamma_i \geq 0$ for each $i \leq n$, $|\gamma| = \gamma_1 + \cdots + \gamma_n$, $\gamma! = \gamma_1! \cdots \gamma_n!$, and $z^\gamma = z_1^{\gamma_1} \cdots z_n^{\gamma_n}$. The space of multipliers on $H^2(\mathbb{B}_n, \omega_k)$ is denoted by $M(\omega_k)$. It consists of all functions $f \in H^2(\mathbb{B}_n, \omega_k)$ such that

$$\|f\|_\infty = \sup\{\|fg\|_2 : g \in H^2(\mathbb{B}_n, \omega_k), \|g\|_2 \leq 1\} < \infty.$$

Notice that $\|f\|_\infty$ coincides with the norm of M_f in $B(H^2(\mathbb{B}_n, \omega_k))$, where M_f is the operator on $H^2(\mathbb{B}_n, \omega_k)$ defined by $M_f(g) := fg$. For more information on $M(\omega_k)$ we refer to [Aro] and [CM]. A standard argument shows that the space of multipliers $M(\omega_k)$ of $H^2(\mathbb{B}_n, \omega_k)$ is unitarily equivalent to $\mathcal{W}(B_1, \dots, B_n)$ (see [Ar2] and [DP3]).

By Proposition 3.3 and Proposition 3.5 we obtain the following.

Corollary 3.9. *Let $I \subset M(\omega_k)$ be a w^*-closed ideal of $M(\omega_k)$ and let $\mathcal{N}_I \subset H^2(\mathbb{B}_n, \omega_k)$ be the orthogonal complement of the image of I in $H^2(\mathbb{B}_n, \omega_k)$. Then the map $\Psi : M(\omega_k)/I \to B(\mathcal{N}_I)$ defined by $\Psi(f(z_1, \dots, z_n) + I) = P_{\mathcal{N}_I} M_{f(z_1,\dots,z_n)}|_{\mathcal{N}_I}$ is a w^*-continuous completely isometric representation.*

Remark 3.10. The techniques used to prove Theorem 3.2 apply to other algebras. The same proof gives that if A is a unital w^*-closed subalgebra of $B(\mathcal{H})$, with the property that for every semi invariant subspace $\mathcal{E} \subset \mathcal{H} \otimes \ell_2$ under $A \otimes I_{\ell_2}$, there exists an isometry $V : \mathcal{E} \to \mathcal{H} \otimes \ell_2$ such that

$$V^*(a \otimes \ell_2) = P_{\mathcal{E}}(a \otimes I_{\ell_2}) P_{\mathcal{E}} V^* \quad \text{for every} \quad a \in A,$$

then $\Psi_J : A/J \to B(\mathcal{N}_J)$ is completely isometric. It follows from arguments in [Mc1] and [Mc2] that some singly generated finite dimensional subalgebras of M_N satisfy this property.

4. Interpolation problems

In this section we present some consequences of Section 3 to interpolation. We will use Theorem 3.2 to deduce Carathéodory's and Nevanlinna-Pick's interpolation theorems on the weighted noncommutative analytic Toeplitz algebra $F^\infty(\omega_\alpha)$.

We need some preliminaries. For each $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{B}_n$, define

$$z_\lambda = \sum_{\alpha \in \mathbb{F}_n^+} \frac{1}{\sqrt{\omega_\alpha}} \bar{\lambda}_\alpha e_\alpha.$$

According to the condition (ω_3) from Section 1, one can see that $\sum_{\alpha \in \mathbb{F}_n^+} \frac{1}{\omega_\alpha} \lambda_\alpha e_\alpha$ is in $\mathcal{F}^2$. Since $\sum_{\alpha \in \mathbb{F}_n^+} \lambda_\alpha e_\alpha \in \mathcal{F}^2$, it is clear that $z_\lambda \in \mathcal{F}^2$ for any $\lambda \in \mathbb{B}_n$. For any $\varphi(W_1, \dots, W_n) := \sum_{\alpha \in \mathbb{F}_n^+} b_\alpha W_\alpha$ in $F^\infty(\omega_\alpha)$ and every $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{B}_n$, we have

$$\begin{aligned} \langle \varphi(W_1, \dots, W_n) e_0, z_\lambda \rangle &= \langle \sum_{\alpha \in \mathbb{F}_n^+} b_\alpha \frac{\sqrt{\omega_\alpha}}{\sqrt{\omega_e}} e_\alpha, \sum_{\alpha \in \mathbb{F}_n^+} \frac{1}{\sqrt{\omega_\alpha}} \bar{\lambda}_\alpha e_\alpha \rangle \\ &= \frac{1}{\sqrt{\omega_e}} \sum_{\alpha \in \mathbb{F}_n^+} b_\alpha \lambda_\alpha = \frac{1}{\sqrt{\omega_e}} \varphi(\lambda). \end{aligned} \tag{4.1}$$

On the other hand, according to (1.8), we infer that $W_i^* z_\lambda = \bar{\lambda}_i z_\lambda$, and hence,

$$\varphi(W_1, \dots, W_n)^* z_\lambda = \overline{\varphi(\lambda)} z_\lambda \tag{4.2}$$

for every $\lambda \in \mathbb{B}_n$ and $\varphi(W_1, \dots, W_n) \in F^\infty(\omega_\alpha)$.

Given $\lambda \in \mathbb{B}_n$, a scalar matrix $A \in M_k(\mathbb{C})$, and an operator matrix $\Phi = [\varphi_{ij}(W_1, \ldots, W_n)]$ in $M_k(F^\infty(\omega_\alpha)) = F^\infty(\omega_\alpha) \otimes_{\min} M_k$, one can use (4.2) to show that $\Phi(\lambda) := [\varphi_{ij}(\lambda)] = A$ if and only if

$$\Phi^*(z_\lambda \otimes x) = z_\lambda \otimes A^* x \quad \text{for any } x \in \ell_2^k. \tag{4.3}$$

Let us remark that, according to (4.1), for every $\lambda \in \mathbb{B}_n$, the linear map $\Phi_\lambda : F^\infty(\omega_\alpha) \to \mathbb{C}$, defined by $\Phi_\lambda(\varphi(W_1, \ldots, W_n)) = \varphi(\lambda)$, is w^*-continuous and multiplicative. Notice that if $\mu_1, \ldots, \mu_N \in \mathbb{B}_n$ are distinct points, then

$$J := \{\varphi(W_1, \ldots, W_n) \in F^\infty(\omega_\alpha) : \varphi(\mu_1) = \varphi(\mu_2) = \cdots = \varphi(\mu_N) = 0\}$$

is a w^*-closed, 2-sided ideal of $F^\infty(\omega_\alpha)$. Using (2.6) and (4.1), we deduce that $\mathcal{N}_J = \text{span}\{z_{\mu_1}, \ldots, z_{\mu_N}\}$.

We shall prove now the Nevanlinna-Pick interpolation theorem on weighted noncommutative analytic Toeplitz algebras.

Theorem 4.1. *Let $\mu_1, \ldots, \mu_N \in \mathbb{B}_n$ be N-distinct points, and let $A_1, \ldots, A_N \in M_k$ be $k \times k$ matrices. Then there exists $\Phi = [\varphi_{ij}(W_1, \ldots, W_n)]$ in $M_k(F^\infty(\omega_\alpha))$ such that $\|\Phi\| \le 1$ and $\Phi(\mu_i) = A_i$ for $i \le N$ if and only if the matrix*

$$[\langle z_{\mu_i}, z_{\mu_j} \rangle (I_k - A_j A_i^*)]_{i,j \le N}$$

is positive semidefinite.

Proof. Suppose that there exists $\Phi = [\varphi_{ij}(W_1, \ldots, W_n)] \in M_k(F^\infty(\omega_\alpha))$ such that $\|\Phi\| \le 1$ and $\Phi(\mu_i) = A_i$ for $i \le N$. Let $T = (P_{\mathcal{N}_J} \otimes I_{\ell_2^k})\Phi|_{\mathcal{N}_J \otimes \ell_2^k}$ be the compression of Φ to $\mathcal{N}_J \otimes \ell_2^k$. Hence, $\langle T^* h, T^* h \rangle \le \langle h, h \rangle$ for every $h = \sum_{i \le N} z_{\mu_i} \otimes x_i$ in $\mathcal{N}_J \otimes \ell_2^k$. According to (4.3), we obtain

$$\sum_{i,j \le N} \langle z_{\mu_i}, z_{\mu_j} \rangle \langle A_j A_i^* x_i, x_j \rangle \le \sum_{i,j \le N} \langle z_{\mu_i}, z_{\mu_j} \rangle \langle x_i, x_j \rangle,$$

and conclude that $[\langle z_{\mu_i}, z_{\mu_j} \rangle (I_k - A_j A_i^*)]_{i,j \le N}$ is positive semidefinite.

Conversely, suppose now that $[\langle z_{\mu_i}, z_{\mu_j} \rangle (I_k - A_j^* A_i)]_{i,j \le N}$ is positive semidefinite. Then, the operator T on $\mathcal{N}_J \otimes \ell_2^k$, defined by $T^*(z_{\mu_i} \otimes x) = z_{\mu_i} \otimes A_i^* x$ for $i \le N$ and $x \in \ell_2^k$, satisfies $\|T\| \le 1$. Similarly to Lemma 5.4 of [APo2], we can find $\Psi = [\psi_{ij}(W_1, \ldots, W_n)] \in M_k(F^\infty(\omega_\alpha))$ such that $T = (P_{\mathcal{N}_J} \otimes I_{\ell_2^k})\Psi|_{\mathcal{N}_J \otimes \ell_2^k}$. By Theorem 3.2, there exists $\Phi \in M_k(F^\infty(\omega_\alpha))$ such that $(P_{\mathcal{N}_J} \otimes I_{\ell_2^k})\Phi|_{\mathcal{N}_J \otimes \ell_2^k} = T$ and $\|\Phi\| = \|T\| \le 1$. Using (4.3), we obtain $\Phi(\mu_i) = A_i$ for every $i \le N$. ∎

Remark 4.2. *Since every operator in* $B(\mathcal{H})$ *can be approximated in the* w^**-topology by matrices,* Theorem 4.1 *remains true for* $A_1, \ldots, A_k \in B(\mathcal{H})$.

Notice that Theorem 4.1 implies Nevanlinna-Pick interpolation theorems in H^∞ (see [N] and [Pic]), in F^∞ (see [APo2] and [DP3]), in the Dirichlet algebra D (see [Q]), and in the space of multipliers $M(\omega_k)$ of the weighted Hardy space $H^2(\mathbb{B}_n, \omega_k)$ (see [Q]).

Now we shall deduce from Theorem 3.2 the Carathéodory interpolation theorem on $F^\infty(\omega_\alpha)$. For each $N \geq 1$, let J_N be the w^*-closed, 2-sided ideal of $F^\infty(\omega_\alpha)$ generated by W_α, $|\alpha| = N$. It is easy to see that $\mathcal{N}_{J_N} = \text{span}\{e_\alpha : |\alpha| < N\}$.

Theorem 4.3. *Let* $p \in \mathcal{P}$*, with* $\deg(p) < N$*. Then*

$$\text{dist}(p(W_1, \ldots, W_n), J_N) = \left\| P_{\mathcal{N}_{J_N}} p(W_1, \ldots, W_n)|_{\mathcal{N}_{J_N}} \right\|.$$

Let us remark that Theorem 4.3 recovers the classical Carathéodory's interpolation theorem on H^∞ (see for example [S]), and on F^∞ (see [Po6]). It also implies Carathéodory interpolation theorems in the Dirichlet algebra and in the space of multipliers $M(\omega_k)$ of the weighted Hardy space $H^2(\mathbb{B}_n, \omega_k)$ (see [Mc1] for $n = 1$).

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