A REMARK ON POSITIVE ISOTROPIC RANDOM VECTORS

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Abstract. A random vector $X = (X_1, \dots, X_n)$ is positive isotropic if $P(X_1 > 0) > 0$, its coordinates are non-negative and identically distributed random variables, and there exists a function $c : R_+^n \to R_+$ so that, for every $a \in R_+^n \setminus \{0\}$, the random variables $\sum a_i X_i$ and $c(a)X_1$ are identically distributed. We study the properties of the function $c(\cdot)$, and prove that $c(\cdot)$ cannot be a norm unless the coordinates of X are equal with probability 1.

1. Introduction

It is well-known that for every $q \in (0,1)$ there exist q-stable random vectors with non-negative coordinates. The classical example is given by the measure μ_q on $R_+^n = \{x \in R^n : \ x_i \geq 0, \ i = 1, ..., n\}$ whose Laplace transform is equal to $\exp(|x_1|^q + ... + |x_n|^q)$ (see [3, 16]). Any random vector $(X_1, ..., X_n)$ with joint distribution equal to μ_q has the property that for every $a = (a_1, \cdots, a_n) \in R_+^n$, the random variable $a_1X_1 + \cdots + a_nX_n$ has the same distribution as $\|a\|_q X_1$, where $\|a\|_q = (|a_1|^q + \cdots |a_n|^q)^{\frac{1}{q}}$. Geometrically, this means that one-dimensional projections of μ_q are equal up to a constant parameter.

In this note, we consider a more general class of random vectors:

Definition 1. *Let $X_1, ..., X_n$ be identically distributed non-negative (with probability 1) random variables with $P(X_1 \neq 0) > 0$. A random vector $X = (X_1, ..., X_n)$ is positive isotropic if there exists a function $c : R_+^n \to R_+$ so that for every $a \in R_+^n$, the random variables $(a, X) = \sum a_i X_i$ and $c(a)X_1$ are identically distributed. We call $c(\cdot)$ the norming functional of X.*

There are several examples of norming functions $c(\cdot)$'s. In addition to the example above, it is known that if $f_1, \cdots, f_n \in L_q$, $0 < q < 1$, are non-negative functions of norm one, then the functional $c(a) = \|a_1 f_1 + \cdots + a_n f_n\|_{L_q}$ is the norming functional of a positive isotropic random vector in R^n (see Proposition 5 below). Let us mention here that positive isotropic random vectors were used to construct isometric embedding of the space L_r in $L_p(L_q)$, where $p \leq r \leq q$ (see [11, 12]).

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If we drop the condition that the X_i's are non-negative and allow a to be an arbitrary vector from R^n, we get the definition of isotropic random vectors introduced by Eaton [2]. The study of isotropic random vectors has a long history and is closely connected with different problems of probability theory and functional analysis (see surveys [5, 9].) One of the main problems in this area is to characterize those functions $c(\cdot)$ that can appear as norming functionals of isotropic vectors. This problem is related to the study of positive definite functions, because $c(\cdot)$ is the norming functional of an isotropic random vector if and only if there exists a function f on R so that $f(c(x))$ is a continuous positive definite function on R^n (i.e. it is the Fourier transform of a finite measure on R^n.) It appears to be quite difficult to check positive definiteness of functions of this type. For example, the question of whether the function $\exp(-\|x\|_q^\beta)$ is positive definite for any $q > 2,\ \beta > 0,\ n > 2$ was posed by Schoenberg in 1938 [13] and remained open for more than fifty years. The answer was given in [8] (for $q = \infty$) and in [4] (for $2 < q < \infty$), where it was shown that these functions were not positive definite. In fact, [8, 7, 14, 15] also show that the norms of the spaces $l_q^n,\ q > 2,\ n > 2$, cannot appear as norming functionals of any isotropic vectors. Note that classical results of P. Levy [6] and Schoenberg [13] imply that the functions $\exp(-\|x\|_q^\beta)$ are positive definite for every $0 < q \le 2$ and $0 < \beta \le q$. Furthermore, these functions are norming functionals of stable vectors.

We show below that the "positive" versions of the latter questions are much easier. Namely, we prove that the only norm which can appear as the function $c(\cdot)$ in Definition 1 is the l_1^n-norm, and only when the coordinates of X are equal to each other with probability 1.

2. Properties of norming functionals

Let us first prove that norming functionals are continuous.

Proposition 1. *The function $c : R_+^n \to R_+$ appearing in Definition 1 has the following properties: (i) for every $a \in R_+^n$ and $k > 0$, $c(ka)=kc(a)$; (ii) $c(a) = 0$ if and only if $a = 0$; (iii) $c(e_i) = 1$ for each $i \le n$, where the e_i's are the standard basis in R^n; and (iv) $c(\cdot)$ is continuous on R_+^n.*

Proof. All properties follow from the condition that $P(X_1 \neq 0) > 0$. The proofs for (i), (ii), and (iii) are immediate. We will check (iv). Let $a^k = (a_1^k, \cdots, a_n^k) \in R_+^n$ be a sequence converging to $a = (a_1, \cdots, a_n) \in R_+^n$. By the continuity of the Laplace transform of the joint distribution of $(X_1, X_2, \cdots, X_n)$, we get that

$$\mathbb{E}e^{-(a^k,X)} = \mathbb{E}e^{-a_1^k X_1 - \cdots - a_n^k X_n} \longrightarrow \mathbb{E}e^{-a_1 X_1 - \cdots - a_n X_n} = \mathbb{E}e^{-(a,X)}.$$

By assumption, we get that

$$\mathbb{E}e^{-c(a^k)X_1} \longrightarrow \mathbb{E}e^{-c(a)X_1}.$$

Since the Laplace transform of the distribution of X_1 is strictly decreasing, we conclude that $c(a^k) \to c(a)$. $\square$

The next fact establishes the connection between positive isotropic vectors and the Laplace transform. Compare Proposition 2 with Proposition 1 [5] to see that the Laplace transform replaces the Fourier transform in the corresponding property of isotropic random vectors.

Proposition 2. *The vector X is positive isotropic with norming functional $c(\cdot)$ if and only if the Laplace transform of the measure μ has the form $\mathcal{L}\mu(x) = f(c(x))$, where $f : R_+ \to R_+$ is a non-constant function.*

Proof. For every $a \in R_+^n,\ a \neq 0$, denote by ν_a the probability measure on R_+ which is the image of the measure μ under the mapping $x \mapsto (a,x)/c(a)$ from R_+^n to R_+.

For every positive number k and every $a \in R_+^n, a \neq 0$, we have

$$\mathcal{L}\mu(ka) = \int_{\mathbb{R}_+^n} \exp(-k(a,x))\ d\mu(x) = \int_{\mathbb{R}_+^n} \exp(-kc(a)(a,x)/c(a))\ d\mu(x) =$$

$$\int_{\mathbb{R}_+} \exp(-kc(a)y)\ d\nu_a(y) = \mathcal{L}\nu_a(c(ka)). \tag{3}$$

Suppose that X is positive isotropic with norming functional $c(\cdot)$. Then, the measures ν_a's are equal to the distribution of the vector X_1. Denote by $f = \mathcal{L}\nu_a$. Then, the representation for $\mathcal{L}\mu$ immediately follows from (3). The function f is non-constant, because $P(X_1 > 0) > 0$ and ν_a cannot be supported in the origin.

Conversely, suppose that for every $k > 0$ and $a \in R_+^n$, $\mathcal{L}\mu(ka) = f(c(ka))$. Since k is arbitrary, it follows from (3) that $\mathcal{L}\nu_a \equiv f$, for every $a \in \mathbb{R}_+^n, a \neq 0$. By the uniqueness theorem for the Laplace transform, we see that all the measures ν_a are equal. Hence, the random vectors $(a,X)/c(a)$ are identically distributed, and the result follows from the fact that $c(e_i) = 1$ for every i. $\square$

A function $\phi : R_+^n \to R_+$ is called completely monotone if ϕ is infinitely differentiable in the interior of R_+^n, and for every choice of non-negative integers $\alpha_1, ..., \alpha_n$,

$$(-1)^{\alpha_1+\ldots+\alpha_n} \frac{\partial^{\alpha_1+\ldots+\alpha_n}\phi}{\partial x_1^{\alpha_1}\ldots\partial x_n^{\alpha_n}} \geq 0,$$

at every point from the interior of R_+^n. The celebrated Bernstein's theorem states that a function ϕ on R_+^n is the Laplace transform of a probability measure on R^n if and only if ϕ is continuous, completely monotone, and $\phi(0) = 1$ (see [1].)

Remark. It follows from the proof of Proposition 2 that the function f corresponding to a positive isotropic vector is the Laplace transform of a measure supported on $[0,\infty)$. Hence, f is a continuous and completely monotone on R_+. On the other hand, Proposition 2 implies that a functional $c(\cdot)$, satisfying the conditions of Proposition 2, is a norming functional of a positive isotropic vector if and only if there exists a non-constant completely monotone function f on R_+ so that $f(c(x))$ is completely monotone on R_+^n.

We will show that a norming functional cannot satisfy the triangle inequality, except for the trivial case where $P(X_1 = X_2 = \cdots = X_n) = 1$.

Proposition 3. *Let $c(\cdot)$ be the norming functional of a positive isotropic vector X. Then, for every $\alpha, \beta > 0$, $c(\alpha e_1 + \beta e_2) \geq \alpha + \beta$. Equality holds only if $P(X_1 = X_2) = 1$.*

Proof. Suppose that $c(\alpha e_1 + \beta e_2) \leq \alpha + \beta$. Since X_1 and X_2 are non-negative, and X_1 has the same distribution as X_2 and $(\alpha X_1 + \beta X_2)/c(\alpha e_1 + \beta e_2)$, we can apply Holder's inequality:

$$\int \exp(-X_1)\; dP = \int \exp\Big(-\frac{\alpha X_1 + \beta X_2}{c(\alpha e_1 + \beta e_2)}\Big)\; dP \leq$$

$$\int \exp\Big(-\frac{\alpha X_1 + \beta X_2}{\alpha + \beta}\Big)\; dP =$$

$$\int \exp\Big(-\frac{\alpha}{\alpha+\beta} X_1\Big) \exp\Big(-\frac{\beta}{\alpha+\beta} X_2\Big)\; dP \leq$$

$$\Big(\int \exp(-X_1)\; dP\Big)^{\frac{\alpha}{\alpha+\beta}} \Big(\int \exp(-X_2)\; dP\Big)^{\frac{\beta}{\alpha+\beta}} = \int exp(-X_1)\; dP.$$

By the equality condition in Holder's inequality, $X_1 = X_2$ with probability 1. But this means that $c(\alpha e_1 + \beta e_2) = \alpha + \beta$. □

Corollary 1. *The only norm that can appear as the norming functional of a positive isotropic random vector in R^n is the norm of the space l_1^n.*

Proof. Suppose that $c(a)$ is a norm. Use Proposition 3 to conclude that all the random variables $X_1, ..., X_n$ are equal to each other with probability 1, and notice that this implies that $c(a) = a_1 + ... + a_n$. □

Another consequence of Proposition 3 is that coordinates of positive isotropic vectors cannot have finite moments of order greater than 1.

Corollary 2. *Let X be a positive isotropic vector in $R^n, n \geq 2$, so that $P(X_1 = X_2 = \cdots = X_n) < 1$. Then, for every $q \geq 1$, the q-th moment of X_1 does not exist; i.e. $\int X_1^q \, dP = \infty$.*

Proof. Let $c(\cdot)$ be the norming functional of X. Suppose that $\int X_1^q \, dP < \infty$. For every vector $a \in \mathbb{R}_+^n$, $a \neq 0$, we have

$$(4) \qquad \Big(\int (a, X)^q \, dP \Big)^{1/q} = (c(a)) \Big(\int X_1^q \, dP \Big)^{1/q}.$$

The left-hand side is the norm of a subspace of L_q, where $q \geq 1$, so $c(a)$ is a norm. By Corollary 1, $c(a) = a_1 + ... + a_n$, and the coordinates are equal with probabiliy 1. This contradicts our assumptions. □

Note that the coordinates of the standard positive q-stable $(0 < q < 1)$ random vector, which were mentioned at the beginning of this note, have finite moments only for orders less than q (see [16]).

If the equality (4) holds with $c(a) = a_1 + ... + a_n$, the coordinates of X are equal for any random vector X with non-negative coordinates (not necessarily positive isotropic).

Proposition 4. *Suppose that $q > 1$ and X is any random vector in R^n with positive coordinates satisfying the equality*

$$(4) \qquad (a_1 + ... + a_n)^q = \int (a, X)^q \, dP$$

for every $a \in R_+^n$. Then, there exist constants $k_1, .., k_n$ so that $P(k_1 X_1 = k_2 X_2 = ... = k_n X_n) = 1$.

Proof. Take the derivatives by a_1 of both sides of (4) and use Holder's inequality (recall that $q > 1$):

$$q(a_1 + \cdots + a_n)^{q-1} = q \int (a, X)^{q-1} X_1 \, dP \leq$$

$$q \Big(\int (a, X)^q \, dP \Big)^{(q-1)/q} \Big(\int X_1^q \, dP \Big)^{1/q} = q(a_1 + \cdots + a_n)^{q-1}.$$

By the equality condition in Holder's inequality, we conclude that there exists a constant k_1 so that $P\big((a, X) = k_1 X_1\big) = 1$. Repeating the same argument for each X_i, $i = 2, ..., n$, we get the desired result. □

To complete this article, we show a large class of positive isotropic vectors whose norming functionals are quasi-norms. This fact is known to specialists.

Proposition 5. *Let $f_1, \dots, f_n \in L_q$, $0 < q < 1$, be non-negative functions with norm 1. Then, the functional $c(a) = \|a_1 f_1 + \dots + a_n f_n\|_{L_q}$ is the norming functional of a positive isotropic random vector in R^n.*

Sketch of the proof. Consider the function $g(a) = exp(-(c(a))^q)$. Looking at the derivatives of $\log(g(a))$ and using logarithmic differentiation one can see that g is completely monotone. Then the result follows from Bernstein's theorem and Proposition 2. □

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