Ph. D. Preliminary Exam in Real Analysis
Syllabus

June 28, 2004

1 Topics

1.1 Metric Spaces

examples of metric spaces; limits, limits of sequences; compact, complete, connected metric spaces; continuous and uniformly continuous functions between metric spaces; Baire category theorem; contraction mapping theorem.

1.2 Topology of the Real Line

A "specialization" of section 1. series of real numbers; intermediate value theorem.

1.3 Differentiation

the usual theorems of calculus, done carefully

1.4 Integration

Development and properties of the Riemann integral

1.5 Sequences and Series of Functions

Pointwise and uniform convergence, power series, interchange of limits results

1.6 Functions of Several Variables

Calculus in several variables; the differential as a linear map.

2 Texts

- *Principles of Mathematical Analysis*, Rudin. McGraw Hill
- *Methods of Real Analysis*, Goldberg. John Wiley
- *Advanced Calculus*, Buck. McGraw Hill
- *Elementary Classical Analysis*, Marsden. Freeman

3 Sample Questions

1. (a) Give the $\epsilon - \delta$ definition of continuity at the point $x = a$ for a function $f : \mathbb{R} \to \mathbb{R}$.

   (b) Using this definition, prove the following: Let $f : \mathbb{R} \to \mathbb{R}$ be continuous at the point $x = a$, and satisfy $f(a) > 0$. Show that there is an open interval (c, d) containing a so that $f(x) > 0$ for all $x \in (c, d)$.

   (c) Let $f, g : \mathbb{R} \to \mathbb{R}$ be functions, with f continuous at $x = a$ and g continuous at $x = f(a)$. Prove that the function $g \circ f : \mathbb{R} \to \mathbb{R}$, defined by $g \circ f(x) = g(f(x))$, is continuous at $x = a$.

2. Let $f_n : [0, 1] \to \mathbb{R}$, $n = 1, 2, \ldots$ be a sequence of functions. and let $f : [0, 1] \to \mathbb{R}$.

   (a) Define: f_n converges to f **pointwise** on $[0, 1]$.

   (b) Define: f_n converges to f **uniformly** on $[0, 1]$.

   (c) Prove or give a counterexample: If f_n converges pointwise to f on $[0, 1]$, then f_n converges uniformly to f on $[0, 1]$.

   (d) Prove that if $f_n, f : [0, 1] \to \mathbb{R}$ are continuous functions and f_n converges to f uniformly, then

$$\lim_{n \to \infty} \int_0^1 f_n(x) dx = \int_0^1 f(x) dx.$$

Note: You may use without proof standard facts about definite integrals, but state the facts you use.

3. Let $a_1, a_2, \ldots, a_n \in \mathbb{R}$ be fixed, and consider the function $f : \mathbb{R}^{\mathbf{n}} \to \mathbb{R}$ defined by

$$f(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} a_i x_i$$

Maximize f subject to the constraint $\sum_{i=1}^{n} x_i^2 = 1$. (Hint: Lagrange multipliers.)

4. (a) Define: The series $\sum_{n=1}^{\infty} a_n = a$.

   (b) Using the definition you gave in part (a), prove that if $-1 < x < 1$, $\sum_{n=0}^{\infty} x^n = \dfrac{1}{1-x}$.

   (c) Prove this version of the ratio test: Suppose that $a_n > 0$ for $n = 1, 2, \ldots$ and

   $$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} < 1.$$

   Then the series $\sum_{n=0}^{\infty} a_n$ converges.

   (d) Prove this version of the root test: Suppose that $a_n > 0$ for $n = 1, 2, \ldots$ and

   $$\lim_{n\to\infty} \sqrt[n]{a_n} < 1.$$

   Then the series $\sum_{n=0}^{\infty} a_n$ converges.

5. Let $f : [0, 1] \to \mathbb{R}$ be a function.

   (a) Give the careful definition of the Riemann integral $\int_0^1 f(x)\,dx$.

   (b) Give an example of a function $f : [0, 1] \to \mathbb{R}$ which is not Riemann integrable.

   (c) Using this definition, prove that $\int_0^1 f(x)\,dx$ exists if f is bounded and continuous except at a finite number of points in $[0, 1]$. *You may **not** assume that a continuous function $f : [0, 1] \to \mathbb{R}$ has a Riemann integral. Your proof should show this fact.*

6. This problem concerns the Baire Category Theorem.

(a) Prove the following version of the Baire Category Theorem: Let (M, d) be a complete metric space, and $U_1, U_2, \ldots$ a sequence of **dense open sets** in M. Prove that

$$\bigcap_{n=1}^{\infty} U_n \neq \emptyset.$$

(b) A subset K of the metric space (M, d) is called a G_δ set if

$$K = \bigcap_{n=1}^{\infty} U_n$$

for some sequence $U_1, U_2, \ldots$ of open subsets of M.
Prove or give a counterexample:

i. The set of irrational numbers is a G_δ subset of $\mathbb{R}$.
ii. The set of rational numbers is a G_δ subset of $\mathbb{R}$.

(c) Prove that there is no function $f : \mathbb{R} \to \mathbb{R}$ such that f is continuous at each rational number and discontinuous at each irrational number.

7. (a) Let $x_n, n = 1, 2, 3, \ldots$ be a sequence of real numbers. Define $\lim_{n\to\infty} x_n = A$.

(b) Suppose that $\lim_{n\to\infty} x_n = A$ and that $\lim_{n\to\infty} y_n = B$. Using the definition in part (a), prove that

$$\lim_{n\to\infty} x_n + y_n = A + B.$$

(c) Suppose that $\lim_{n\to\infty} x_n = A$ and that $A \neq 0$. Prove that

$$\lim_{n\to\infty} \frac{1}{x_n} = \frac{1}{A}$$

8. Suppose that I is a non-empty interval of real numbers and $f : I \to \mathbb{R}$ is a continuous, real valued function.
Prove or give a counterexample to each of the following statements.

(a) If I is an open interval, then f is a bounded function on I.

(b) If I is an arbitrary bounded interval, then f assumes its maximum value on I.

(c) If I is a closed bounded interval, then f assumes its maximum value on I.

9. This is a collection of problems concerning integrals.

   (a) Give an example of a function $f : [0,1] \to \mathbb{R}$ which is not Riemann integrable.

   (b) Using any reasonable definition of the definite integral, and **not the Fundamental Theorem of Calculus**, show that

   $$\frac{49}{68} < \int_0^1 \frac{1}{x^4+1}\,dx < \frac{33}{34}.$$

   Hint: Begin with a sketch.

   (c) Let $f : [0,1] \to \mathbb{R}$ be a continuous function. Using any reasonable definition of the definite integral, show that

   $$|\int_0^1 f(x)\,dx| \leq \int_0^1 |f(x)|\,dx.$$

10. This problem concerns homeomorphisms:

    (a) Define: f is a **homeomorphism** between the metric spaces X_1 and X_2.

    (b) Give an example of two metric spaces X_1 and X_2 and a function and $f : X_1 \to X_2$ which is 1-1, onto and continuous, yet is not a homeomorphism. Prove your assertion.

    (c) Prove that if X_1 is a compact metric space and $f : X_1 \to X_2$ is 1-1, onto and continuous, then f is a homeomorphism.

11. Let $f : \mathbb{R} \to \mathbb{R}$ be 1-1, onto and continuous. Prove that f is a homeomorphism.

12. (a) State the Implicit Function Theorem.

    (b) Let $G(x,y) = x^4 - y^4$. Observe that $G(0,0) = 0$. Is y a function of x locally near the point $(0,0)$ on the set $\{(x,y) : G(x,y) = 0\}$?

(c) Same as part (b), but now $G(x, y) = x^4 - y^4 + y$.

13. Suppose that X and Y are Banach spaces and let $T : X \to Y$ be linear, 1-1, onto and continuous. Prove that T is a homeomorphism (i.e., T^{-1} is continuous).

14. Define $T : \mathbb{R}^n \to \mathbb{R}^n$ by $Tx = Ax$, where A is the $n \times n$ matrix $A = \begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$. Recall that ℓ_p^n is the the vector space $\mathbb{R}^n$ with the $\|\bullet\|_p$ −norm. Find the norm of the following linear operators:

    (a) $T : \ell_2^n \to \ell_2^n$,

    (b) $T : \ell_1^n \to \ell_1^n$,

    (c) $T : \ell_\infty^n \to \ell_\infty^n$.,

    (d) $T : \ell_p^n \to \ell_p^n$, where $1 \leq p < \infty$.

15. Prove that the dual space of c_0 is ℓ_1.

16. Prove that the linear span (not a closed linear span) of an infinite sequence of a linearly independent vectors in a normed space is not complete.

17. Let $p \geq 1$ and $q \geq 1$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$ and let (a_n) be a sequence of real numbers. Prove that if $\sum |a_n b_n| < \infty$ for every $(b_n) \in \ell_q$, then $(a_n) \in \ell_p$.