Ph.D. Preliminary Examination in Complex Analysis Fall 2003

May 4, 2005

Instructions. Answer questions 1-4, and **any two** of 5, 6 and 7.

Notation. $\mathbb{C}$ denotes the set of complex numbers. $D = \{z \in \mathbb{C} : |z| \leq 1\}$.

1. (a) Find the Laurent series for the function

$$\frac{1}{z(z-2)}$$

valid in the punctured disk $\{z : 0 < |z| < 2\}$.

(b) Find

$$\int_C \frac{1}{z(z-2)} dz$$

where C is the unit circle, oriented counterclockwise.

2. Compute i^i.

3. This problem concerns analytic functions.

(a) Define: The function f is *analytic* at the point $z_0 \in \mathbb{C}$.

(b) Define: The function f is *analytic* on the connected open set U.

(c) Let $f(z) = |z|^2$. Is f analytic at any point $z_0 \in \mathbb{C}$? Prove your answer.

4. Give an example of a function f with the following properties.

- f maps D onto D
- f is analytic on D.
- $f(1/2) = 0$.
- $f(1) = i$.
- f^{-1} exists on D. Give an explicit formula for f^{-1}.

5. Using a contour integral, find

$$\int_{-\infty}^{\infty} \frac{x^2}{x^4+16} dx.$$

Carefully show all of your work, including estimates, how you evaluate integrals, etc.

6. This problem concerns Rouche's theorem.

   (a) State Rouche's theorem.

   (b) How many zeros (including multiplicity) does the function $P(z) = z^5 + 5z^3 + 7$ have inside the unit circle $|z| = 1$? inside the circle $|z| = 2$?

   (c) Let $f(z) = z + z^5$. Using Rouche's theorem, give a careful proof of the following:

   $$\text{If } |w| < .25 \text{ there is a } z,\ |z| < .8 \text{ with } f(z) = w.$$

7. (a) Suppose that $f(z)$ is analytic throughout an open neighborhood $U = \{z : |z - z_0| < \varepsilon\}$ of the point z_0. Also suppose that

   $$|f(z)| \leq |f(z_0)|$$

   for all $z \in U$. **Prove** that $f(z) = f(z_0)$ for all $z \in U$.

   (b) State the maximum modulus principle. Use the result in part (a) to *sketch* a proof of this result.

   (c) Using the hint below, prove the Schwarz' Lemma:
   Suppose that f is analytic on the open unit disk $U = \{z : |z| < 1\}$. Suppose also that $f(0) = 0$ and $|f(z)| \leq 1$ on U.
   Then (i) $|f(z)| \leq |z|$; and, (ii) $|f'(0)| \leq 1$ on U.

   Hint: Apply the maximum modulus principle to the function

   $$g(z) = \begin{cases} \dfrac{f(z)}{z} & if \quad z \neq 0 \\ f'(0) & if \quad z = 0 \end{cases}$$

   You should argue that g is analytic on U.