# Ph.D. Preliminary Examination in Complex Analysis Fall 1998

**Instructions.** Answer questions 1, 2 and 3, and **any two** of 4, 5 and 6.

**Notation.** $\mathbf{C}$ denotes the set of complex numbers.

1. (a) Find the Laurent series for the function

$$\frac{1}{z(z-2)}$$

valid in the punctured disk $\{z : 0 < |z| < 2\}$.

(b) Find

$$\int_C \frac{1}{z(z-2)} dz$$

where $C$ is the unit circle, oriented counterclockwise.

2. Compute $i^i$.

3. This problem concerns analytic functions.

(a) Define: The function $f$ is *analytic* at the point $z_0 \in \mathbf{C}$.

(b) Define: The function $f$ is *analytic* on the connected open set $U$.

(c) Let $f(z) = |z|^2$. Is $f$ analytic at any point $z_0 \in \mathbf{C}$? Prove your answer.

4. Using a contour integral, find

$$\int_{-\infty}^{\infty} \frac{x^2}{x^4+1} dx.$$

Carefully show all of your work, including estimates, how you evaluate integrals, etc.

5. This problem concerns Rouche's theorem.

(a) State Rouche's theorem.

(b) Let $f(z) = z + z^5$. Using Rouche's theorem, give a careful proof of the following: If $|w| < .25$ there is a $z$, $|z| < .8$ with $f(z) = w$.

6. (a) Suppose that $f(z)$ is analytic throughout an open neighborhood $U = \{z : |z - z_0| < \varepsilon\}$ of the point $z_0$. Also suppose that $|f(z)| \leq |f(z_0)|$ for all $z \in U$. **Prove** that $f(z) = f(z_0)$ for all $z \in U$.

   (b) State the maximum modulus principle. Use the result in part (a) to *sketch* a proof of this result.